Fig. 13.4 shows the vertical stress distribution on a vertical line at distance r from the axis of loading. The vertical stress first increases, attains a maximum value, and then decreases. It can be shown (see example 13.2) that the maximum value of σ_z on a vertical line is obtained at the point of intersection of the vertical plane with a radial line at $\beta = 39° 15'$ through the point load, as shown in Fig. 13.4. The corresponding value of $\frac{r}{z} = 0.817$

or $$z = \frac{r}{0.817} = \frac{1}{0.817} = 1.225$$

for which $$K_B = 0.1332$$

Hence $$(\sigma_z)_{max} = \frac{0.1332\ Q}{(1.225)^2} = \mathbf{0.0888\ Q.}$$

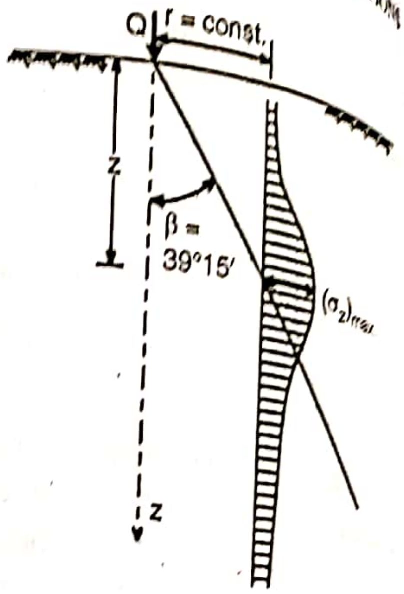

FIG. 13.4. σ_z DISTRIBUTION ON VERTICAL LINE.

Example 13.1. *Find the intensity of vertical pressure and horizontal shear stress at a point 4 m directly below a 20 kN point load acting at a horizontal ground surface. What will be vertical pressure and shear stress at a point 2 m horizontally away from the axis of loading but at the same depth of 4 m?*

Solution : (*a*) $r = 0$; $z = 4$ m ; $Q = 20$ kN.

From Eq. 13.2, $$\sigma_z = \frac{3Q}{2\pi z^2}\left[\frac{1}{1+\left(\frac{r}{z}\right)^2}\right]^{5/2} = \frac{3 \times 20}{2 \times \pi \times (4)^2} = \mathbf{0.597\ kN/m^2}$$

From Eq. 13.3, $$\tau_{rz} = \frac{3Q}{2\pi}\frac{r z^2}{(r^2+z^2)^{5/2}} = \frac{3\,Q\,r}{2\pi z^3}\left[\frac{1}{1+\left(\frac{r}{z}\right)^2}\right]^{5/2} = 0 \text{ (since } r=0).$$

Alternatively, From Table 13.1, $K_B\left(\text{for } \frac{r}{z} = 0\right) = 0.4775$

$\therefore$ $$\sigma_z = K_B\frac{Q}{z^2} = \frac{0.4775 \times 20}{(4)^2} = 0.597 \text{ kN/m}^2.$$

(*b*) $$r = 2 \text{ m};\ z = 4 \text{ m} \ \therefore\ \frac{r}{z} = 0.5$$

$\therefore$ $$\sigma_z = \frac{3 \times 20}{2\pi\,(4)^2}\left[\frac{1}{1+(0.5)^2}\right]^{5/2} = \mathbf{0.342\ kN/m^2}$$

$$\tau_{rz} = \frac{3 \times 2 \times 20}{2\pi\,(4)^2}\left[\frac{1}{1+(0.5)^2}\right]^{5/2} = \mathbf{0.171\ kN/m^2}$$

Alternatively, $K_B\left(\text{for } \frac{r}{z} = 0.5\right) = 0.2733$ $\quad\therefore\ \sigma_z = K_B\frac{Q}{z^2} = \frac{0.2733 \times 20}{(4)^2} = 0.342 \text{ kN/m}^2$

and $$\tau_{rz} = \sigma_z \cdot \frac{r}{z} = 0.342 \times \frac{2}{4} = 0.171 \text{ kN/m}^2$$

Example 13.2. *Prove that the maximum vertical stress on a vertical line at a constant radial distance r from the axis of a vertical load is induced at the point of intersection of the vertical line with a radial line at β = 39° 15′ from the point of application of concentrated load. What will be the value of shear stress at the point ? Hence, or otherwise, find the maximum vertical stress on a line situated at r = 2 m from the axis of a concentrated load of value 20 kN.*

Solution. (Refer Fig. 13.4)

We have

$$\sigma_z = \frac{3Q}{2\pi}\frac{z^3}{(z^2+r^2)^{5/2}} = \frac{3Q}{2\pi z^2}\left[\frac{1}{1+\left(\frac{r}{z}\right)^2}\right]^{5/2} \quad \text{...(13.2)}$$

For the maximum value of σ_z (where r is constant), differentiate Eq. 13.2. with respect to z and equate it to zero.

$$\frac{d\sigma_z}{dz} = \frac{3Q}{2\pi}\left[\frac{3z^2(z^2+r^2)^{5/2} - z^3 \times \frac{5}{2}(z^2+r^2)^{3/2}\, 2z}{(z^2+r^2)^{5/2}}\right] = 0$$

$$3z^2(r^2+z^2) - 5z^4 = 0, \quad \text{from which} \quad z = (\sqrt{3/2})\, r = 1.225\, r \quad \text{...(13.12)}$$

$$\therefore \quad \frac{r}{z} = \sqrt{\frac{2}{3}} = \frac{1}{1.225} = 0.817 = \tan\beta \quad \therefore \quad \beta = 39°\ 15'$$

$\therefore$ Substituting the value of $\frac{r}{z} = \sqrt{\frac{2}{3}}$ and $z = \sqrt{\frac{3}{2}}\, r$ in Eq. 13.12, we get

$$(\sigma_z)_{max} = \frac{3Q}{2\pi}\frac{1}{(\sqrt{\frac{3}{2}})^2}\left[\frac{1}{1+\frac{2}{3}}\right]^{5/2} = \frac{Q}{\pi r^2} \times \frac{1}{(1+\frac{2}{3})^{5/2}} = 0.0888\,\frac{Q}{r^2} \quad \text{...(13.13)}$$

$$\tau_{rz} = \frac{3Q}{2\pi}\frac{r z^2}{(r^2+z^2)^{5/2}} = \frac{3Q}{2\pi}\frac{r}{z^3}\left[\frac{1}{1+\left(\frac{r}{z}\right)^2}\right]^{5/2} = (\sigma_z)_{max}\frac{r}{z} = \left(0.0888\,\frac{Q}{r^2}\right) \times 0.817 = 0.0725\,\frac{Q}{r^2} \quad \text{...(13.14)}$$

When $r = 2$ m and $Q = 20$ kN, $(\sigma_z)_{max} = 0.0888 \times \frac{20}{4} = 0.444$ kN/m²

which occurs at $z = 1.225\, r = 2.45$ m

Also $\tau_{rz} = 0.0725\,\frac{Q}{r^2} = \frac{0.0725 \times 20}{4} = 0.362$ kN/m²

13.4. VERTICAL PRESSURE UNDER A UNIFORMLY LOADED CIRCULAR AREA

The Boussinesq equation for the vertical stress due to a single concentrated load can now be extended to find the vertical pressure on any point on the vertical axis passing through the centre of a uniformly loaded circular area. Fig. 13.5 shows a uniformly loaded circular area of radius a and load intensity q per unit area. Assume the soil as an elastic, isotropic, semi-infinite mass.

Consider an elementary ring of radius r and width δr on the loaded area. If the elementary ring is further divided into small parts, each of area δA, the load on each elementary area will be $q\ \delta A$. This load may be considered as a point load. Hence the vertical pressure at point P, situated at depth z on the vertical axis through the centre of the area, is evidently given by Eq. 13.2 .

$$\delta\sigma_z = \frac{3\,(q\,.\,\delta A)}{2\pi}\,\frac{z^3}{(r^2+z^2)^{5/2}}$$

Integrating over the entire ring of radius r, the vertical stress $\Delta\sigma_z$ is given by

$$\Delta\sigma_z = \frac{3q}{2\pi}\,(\Sigma\delta A)\,\frac{z^3}{(r^2+z^2)^{5/2}} = \frac{3q\,(2\pi r\delta r)}{2\pi}\,\frac{z^3}{(r^2+z^2)^{5/2}} = 3qr\,\delta r\,\frac{z^3}{(r^2+z^2)^{5/2}}$$

The total vertical pressure σ_z due to the entire loaded area is given by integrating the above expression between the limits $r=0$ to $r=a$.

$$\therefore \quad \sigma_z = 3\,q\,z^3\int_0^a \frac{r\,dr}{(r^2+z^2)^{5/2}}$$

Put $r^2+z^2=n^2$, so that $r\,dr = n\,dn$

Limits : $\begin{cases}\text{when } r=0,\ n=z \\ \text{when } r=a,\ n=(a^2+z^2)^{1/2}\end{cases}$

$$\therefore \quad \sigma_z = 3\,q\,z^3\int_z^{(a^2+z^2)^{1/2}} \frac{dn}{n^4}$$

$$= q\,z^3\left[\frac{1}{z^3}-\frac{1}{(a^2+z^2)^{3/2}}\right]$$

$$\therefore \quad \sigma_z = q\left[1-\left\{\frac{1}{1+\left(\frac{a}{z}\right)^2}\right\}^{3/2}\right] \qquad \text{...(13.15)}$$

or $\quad \sigma_z = K_B\,.\,q \qquad$ ...(13.16)

where $\quad K_B$ = Boussineq influence factor for uniformly distributed circular load

$$= 1-\left\{\frac{1}{1+\left(\frac{a}{z}\right)^2}\right\}^{3/2} \qquad \text{...(13.16 } a\text{)}$$

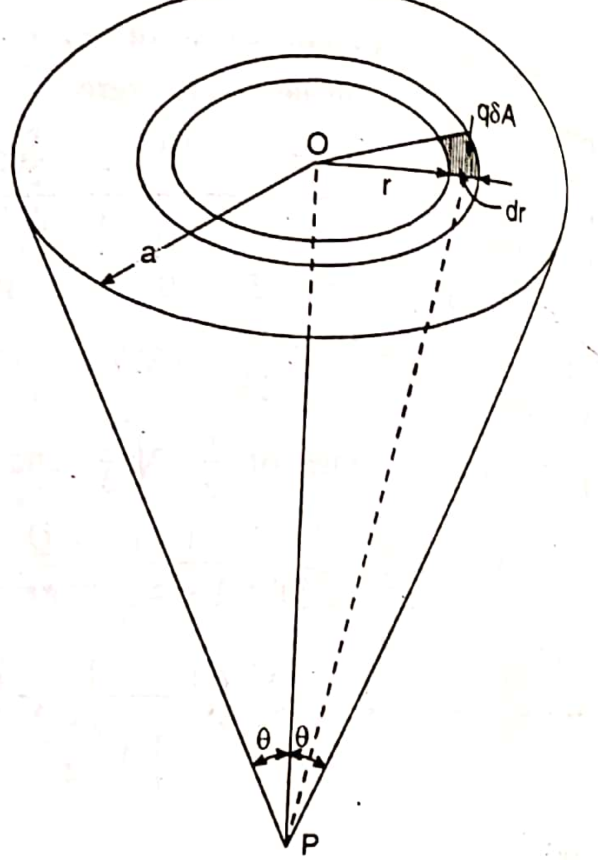

FIG. 13.5. UNIFORMLY DISTRIBUTED LOAD OVER CIRCULAR AREA.

Table 13.5. given the value of the influence factors for various values of a/z . The vertical pressure at a given depth on the vertical axis through the centre of the circular loaded area can be found by multiplying the influence factor by the load intensity q. For the vertical pressure at any other point not situated under the centre of the circular load, reference is drawn to article 5 of chapter 14.

If θ is the angle which the line joining the point P makes with the outer edge of the loading, Eq. 13.15 reduces to

$$\sigma_z = q\,[1-\cos^3\theta\,] \qquad \text{...(13.17)}$$

TABLE 13.5 INFLUENCE FACTORS FOR VERTICAL PRESSURE UNDER CENTRE OF UNIFORMLY LOADED CIRCULAR AREA

$\frac{a}{z}$	K_B	$\frac{a}{z}$	K_B	$\frac{a}{z}$	K_B
0.00	0.0000	1.00	0.6465	2.0	0.9106
0.05	0.0037	1.05	0.6720	2.5	0.9488
0.10	0.0148	1.10	0.6956	3.0	0.9684
0.15	0.0328	1.15	0.7175	3.5	0.9793
0.20	0.0571	1.20	0.7376	4.0	0.9857
0.25	0.0869	1.25	0.7562	4.5	0.9898
0.30	0.1213	1.30	0.7733	5.0	0.9925
0.35	0.1592	1.35	0.7891	5.5	0.9943
0.40	0.1996	1.40	0.8036	6.0	0.9956
0.45	0.2417	1.45	0.8170	6.5	0.9965
0.50	0.2845	1.50	0.8293	7.0	0.9972
0.55	0.3273	1.55	0.8407	8.0	0.9981
0.60	0.3695	1.60	0.8511	9.0	0.9987
0.65	0.4106	1.65	0.8608	10	0.9990
0.70	0.4052	1.70	0.8697	12	0.9994
0.75	0.4880	1.75	0.8779	14	0.9996
0.80	0.5239	1.80	0.8855	16	0.9998
0.85	0.5577	1.85	0.8925	20	0.9999
0.90	0.5893	1.90	0.8990	100	1.0000
0.95	0.6189	1.95	0.9050	∞	1.0000

Fig 13.6. shows a family of isobars under a uniformly loaded circular area, first presented by Jurgenson (1934).

With the help of this diagram, the vertical pressure of various points below a circular loaded area can be conveniently determined.

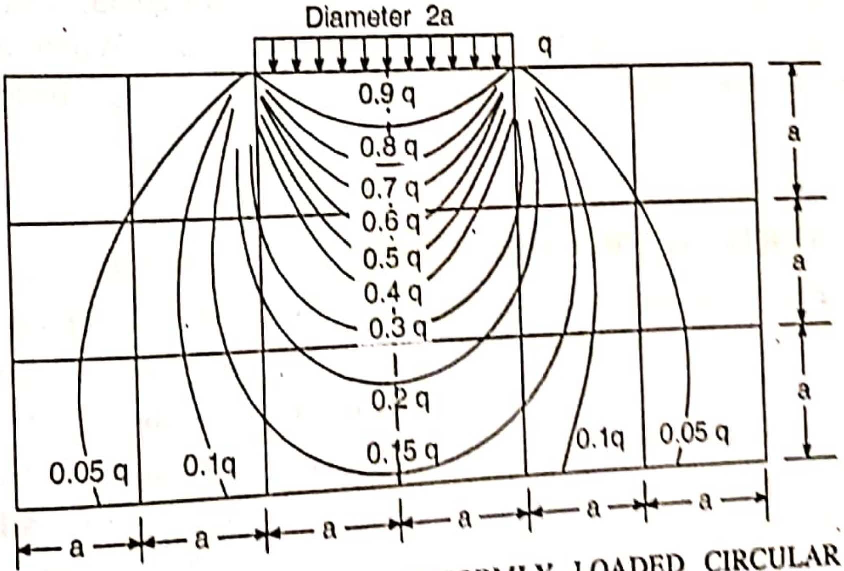

FIG. 13.6. ISOBARS UNDER A UNIFORMLY LOADED CIRCULAR AREA.

13.5. VERTICAL PRESSURE DUE TO A LINE LOAD

Let us consider an infinitely long line load of intensity q' per unit length, acting on the surface of a semi-infinite elastic medium. Let the y-axis be directed along the direction of the line load, as shown in Fig. 13.7. Let us find the expression for the vertical stress at any point P having co-ordinates (x, y, z).

The radial distance of point $P = r = (x^2 + y^2)^{1/2}$

The polar distance of point $P = R = (r^2+z^2)^{1/2} = (x^2+y^2+z^2)^{1/2}$

Consider a small length δy along the line load. The elementary load in this length will be equal to $q' \cdot \delta y$, which can be considered to be a concentrated load. Hence the vertical stress $\Delta \sigma_z$ due to this elementary load is given by

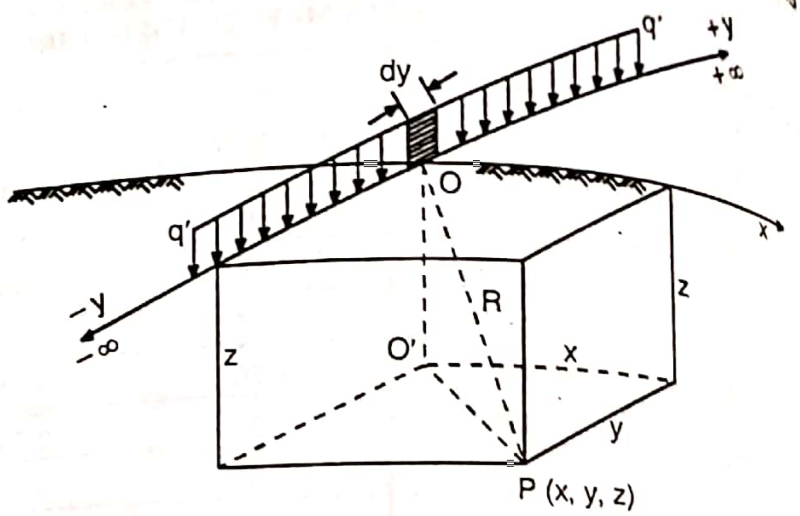

FIG. 13.7. LINE LOAD

$$\Delta\sigma_z = \frac{3(q'\delta y)z^3}{2\pi R^5} = \frac{3\, q'\delta y\, z^3}{2\pi(x^2+y^2+z^2)^{5/2}}$$

$$\therefore \quad \sigma_z = \int_{-\infty}^{+\infty} \frac{3\, q' z^3 dy}{2\pi(x^2+y^2+z^2)^{5/2}} = 2\int_0^{\infty} \frac{3\, q' z^3 dy}{2\pi(x^2+y^2+z^2)^{5/2}}$$

or

$$\sigma_z = \frac{2\, q' z^3}{\pi(x^2+z^2)^2} = \frac{2\, q'}{\pi z} \frac{1}{\left[1+\left(\frac{x}{z}\right)^2\right]^2} \qquad \ldots(13.18)$$

In the above expression x and z are constants for a given position of a point P and the only variable is y. Also, x is the horizontal distance of point P from the line load, in a direction perpendicular to the line load. When the point P is situated vertically below the line load, at a depth z, we have $x = 0$, and hence the vertical stress is given by

$$\sigma_z = \frac{2\, q'}{\pi z} \qquad \ldots(13.19)$$

13.6. VERTICAL PRESSURE UNDER STRIP LOAD

Fig. 13.8 shows an infinite strip of width B, loaded with uniformly distributed load intensity q per unit area. Let us find the vertical pressure at a point P situated below a depth z, on a vertical axis passing through the centre of the strip.

Consider a strip load of width dx, at distance x from the centre. The elementary line load intensity along this elementary strip of width dx will be $q.dx$. The vertical pressure at P due to this elementary line load is given by Eq. 13.18,

$$\Delta\sigma_z = \frac{2(q\, dx)}{\pi z} \cdot \frac{1}{\left[1+\left(\frac{x}{z}\right)^2\right]^2}$$

Total vertical pressure due to the whole strip load is given by

20 equal area units, each area unit will exert a pressure equal to 0.005 q intensity at a depth of 5 cm.

Let the radius of second concentric circle be equal to r_2 cm. By extending the twenty radial lines, the space between the two concentric circles is again divided into 20 equal area units ; $A_1 A_2 B_2 B_1$ is one such area unit. The vertical pressure, at the centre, due to each of these area units is to be of intensity 0.005 q. Therefore, the total pressure due to area units $O A_1 B_1$ and $A_1 A_2 B_2 B_1$ at depth $z = 5$ cm below the centre is $2 \times 0.005\, q$. Hence from Eq. 13.15 :

$$\text{Vertical pressure due to } OA_2 B_2 = \frac{q}{20}\left[1 - \left\{\frac{1}{1+\left(\frac{r_2}{z}\right)^2}\right\}^{3/2}\right] = 2 \times 0.005\, q$$

Substituting $z = 5$ cm, we get $r_2 = 2.00$ cm from the above relation. Similarly, the radii of 3rd, 4th 5th 6th 7th 8th, 9th circles can be calculated, as tabulated in Table 13.8. The radius of 10th circle is given by the following governing equation :

$$\frac{q}{20}\left[1 - \left\{\frac{1}{1+\left(\frac{r_{10}}{z}\right)^2}\right\}^{3/2}\right] = 10 \times 0.005\, q = \frac{q}{20}$$

From the above r_{10} = infinity.

TABLE 13.8 RADII OF CONCENTRIC CIRCLES FOR INFLUENCE CHART
(z=5 cm; i_f = 0.005; each circle divided into 20 parts)

Number of circles	1	2	3	4	5	6	7	8	9	$9\frac{1}{2}$	10
Radius (cm)	1.35	2.00	2.59	3.18	3.83	4.59	5.54	6.94	9.54	12.62	∞

Fig. 13.15. shows the influence chart drawn on the basis of Table 13.8.

To use the chart for determining the vertical stress at any point under the loaded area, the plan of the loaded area is first drawn on a tracing paper to such a scale that the length AB (=5 cm) drawn on the chart represents the depth to the point at which pressure is required. For example, if the pressure is to be found at a depth of 5 m, the scale of plan will be 5 cm = 5 m, or 1 cm = 1 m. The plan of the loaded area is then so placed over the chart that the point below which pressure is required coincides with the centre of the chart . The point below which pressure is required may lie within or outside the loaded area. The total number of area units (including the fractions) covered by the plan of the loaded area is counted. The vertical pressure is then calculated from the relation :

$\sigma_z = 0.005\, q \times N_A$ (where N_A = number of area units under the loaded area). ...(13.31)

Example 13.3. *A rectangular area 2 m × 4 m carries a uniform load of 80 kN/m² at the ground surface. Find the vertical pressures at 5 m below the centre and corner of the loaded area.*

Solution. (a) For the point under the centre of the area, there will be influence of four rectangles of size 1 m × 2 m, having a common corner at the centre of the loaded rectangle.

Hence $a = 1$ m ; $b = 2$ m ; $m = \frac{a}{z} = \frac{1}{5} = 0.2$; $n = \frac{b}{z} = \frac{2}{5} = 0.4$

$\therefore$ K_{B1} (for one quadrant) $= 0.0328$ $\therefore$ $\sigma_z = 4\,qK_{B1} = 4 \times 80 \times 0.0328 =$ **10.5 kN/m²**

(*b*) For the point under the corner of rectangle ;

$a = 2$ m ; $b = 4$ m $\therefore$ $m = \frac{2}{5} = 0.4$; $n = \frac{4}{5} = 0.8$

$\therefore$ $K_B = 0.0931$ $\therefore$ $\sigma_z = q\,K_B = 80 \times 0.0931 =$ **7.45 kN/m²**

Example 13.4. *Solve example 13.3 by the equivalent load method.*

Solution. Divide loaded area into four equal rectangles of size 1 m × 2 m. Each area will represent a point load $Q' = 1 \times 2 \times 80 = 160$ kN acting at its centroid.

(*a*) ***For the point under the centre*** : The influence of each area unit will be equal

$$r' = \sqrt{1 + (0.5)^2} = 1.117$$

$$\therefore \quad \frac{r'}{z} = \frac{1.117}{5} = 0.223 \quad \therefore \quad K_B = 0.4247$$

$$\therefore \quad \sigma_z = \frac{Q'}{z^2}\,\Sigma K_B = \frac{160 \times 4 \times 0.4247}{5 \times 5}$$

$$= \mathbf{10.87\ kN/m^2}$$

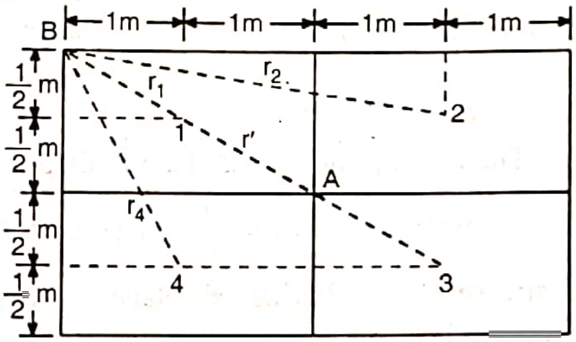

FIG. 13.16.

By exact method (example 13.3), $\sigma_z = 10.5$ kN/m²

$$\therefore \quad \%\ \text{error} = \frac{10.87 - 10.5}{10.5} = 3.5\%$$

(*b*) ***For the point under corner B*** : The influence of each area unit will be different. Let r_1, r_2, r_3, r_4 be the radial distance of centroids of each unit from *B*.

The corresponding values of r/z and K_B are as under :

Area unit	*r*	*r/z*	K_B
1	1.117	0.223	0.4247
2	3.040	0.608	0.2174
3	3.360	0.672	0.1880
4	1.800	0.360	0.3521
			$\Sigma K_B = 1.1822$

$$\therefore \quad \sigma_z = \frac{Q'}{z^2}\,\Sigma K_B = \frac{160 \times 1.1822}{25} = \mathbf{7.57\ kN/m^2}$$

But by exact method (example 13.3), $\sigma_z = 7.45$ kN/m²

$$\therefore \quad \%\ \text{error} = \frac{7.57 - 7.45}{7.45} = 1.56\ \%$$

Example 13.5 *Solve example 13.3. using Newmark's influence chart.*

Solution. $z = 5$ m $\therefore$ Hence the scale of the plan will be

$$AB (= 3 \text{ cm}) = 3 \text{ m} \quad \text{or} \quad 1 \text{ cm} = 1 \text{ m}$$

(a) The plan of the rectangular area is drawn to the scale of 1 cm = 1 m, and oriented on the chart in such a way that its centroid is over the centre of the diagram.

Number of area units under the rectangle $= N_A = 25.5$ units

$\therefore \sigma_z$ under centre of area $= 0.005 \times q\, N_A = 0.005 \times 80 \times 25.5 = 10.2 \text{ kN/m}^2$.

(b) The plan of the rectangular area is then oriented in such a way that one of its corner is above the centre of chart. Then $N_A = 18.5$ units

σ_z under corner of area $= 0.005 \times 80 \times 18.5 = 7.4 \text{ kN/m}^2$.

WESTERGAARD'S ANALYSIS

Determination of pre-consolidation pressure. To find the preconsolidation pressure, an undisturbed sample of clay is consoli-dated in the laboratory and the pressure voids-ratio relationship is plotted on a semilog plot, as shown in Fig. 15.4.

The initial portion of the curve is flat and resembles the recompression curve of a remoulded specimen. The lower portion of the curve, which is a straight line, is the laboratory virgin curve. The approximate value of the pre-consolidation pressure σ_p' may be determined by the following empirical method of A. Casagrande (1936). The point *A* of maximum curvature (minimum radius) is selected and horizontal line *AB* is drawn. A tangent *AC* is drawn to the curve and the bisector *AD*, bisecting angle *BAC* is drawn. The straight portion of the virgin curve is extended back to meet the bisector *AD* in *P*. The point *P* corresponds to the pre-consolidation pressure σ_p'

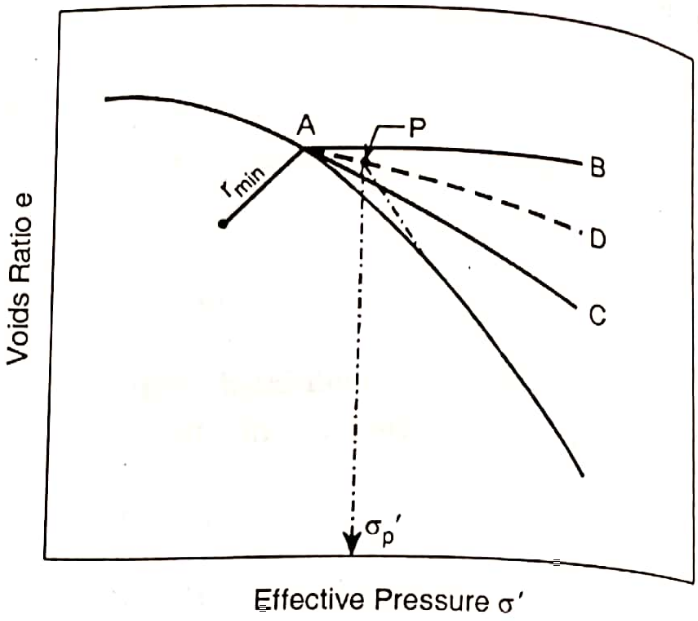

FIG. 15.4. PRE-CONSOLIDATION PRESSURE.

15.5. TERZAGHI'S THEORY OF ONE-DIMENSIONAL CONSOLIDATION

The theoretical concept of the consolidation process was developed by Terzaghi (1923). In the development of the mathematical statement of the consolidation process, the following simplifying assumptions are made : (*i*) The soil is homogeneous and fully saturated. (*ii*) Soil particles and water are incompressible. (*iii*) The deformation of the soil is due entirely to change in volume. (*iv*) Darcy's law for the velocity of flow of water through soil is perfectly valid. (*v*) Coefficient of permeability is constant during consolidation. (*vi*) Load is applied in one direction only and deformation occurs only in the direction of the load application, *i.e.* the soil is restrained against lateral deformation. (*vii*) Excess pore water drains out only in the vertical direction. (*viii*) The boundary is a free surface offering no resistance to the flow of water from the soil. (*ix*) The change in thickness of the layer during consolidation is insignificant (*x*) The time lag in consolidation is due entirely to the permeability of soil, and thus, the secondary consolidation is disregarded.

Fig. 15.5 (*a*) shows a clay layer, of thickness *H*, sandwiched between two layers of sand which serves as drainage faces. When the layer is subjected to a pressure increment $\Delta\sigma$, excess hydrostatic pressure is set up in the clay layer. At the time t_0, the instant of pressure application, whole of the consolidating pressure $\Delta\sigma$ is carried by the pore water so that the initial excess hydrostatic pressure $\bar{u}_0$ is equal to $\Delta\sigma$, and is represented by a straight line $\bar{u} = \Delta\sigma$ on the pressure distribution diagram. The straight line *CED* joining the water levels in the piezometric tubes represent this distribution. As water starts escaping into the sand, the excess hydrostatic pressure at the pervious boundaries drops to zero and remains so at all times. After a very great time t_f, the whole of the excess hydrostatic

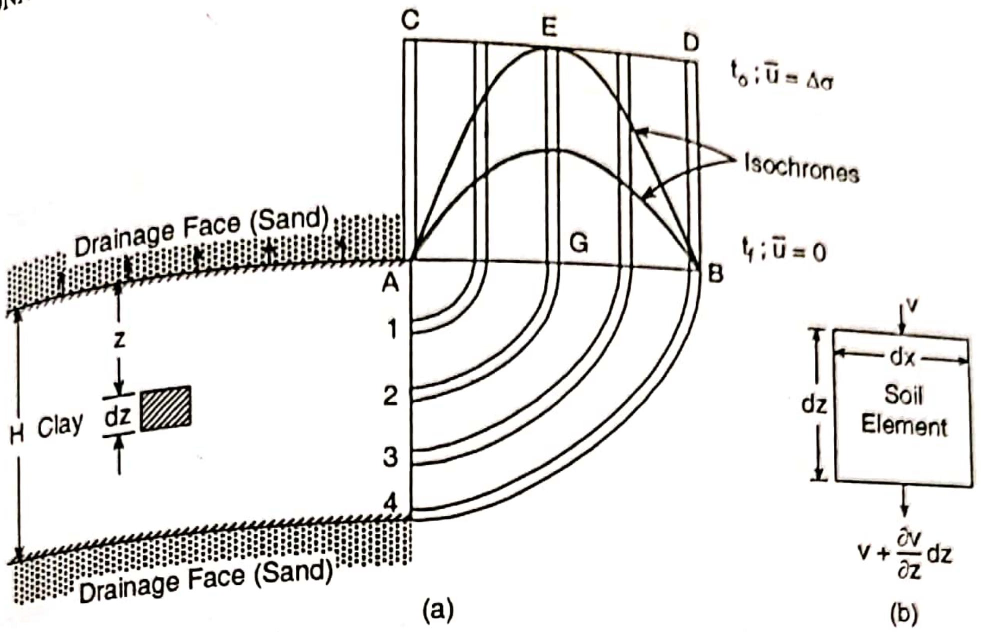

FIG. 15.5. ONE DIMENSIONAL CONSOLIDATION.

pressure is dissipated so that $\bar{u} = 0$, represented by line *AGB*. At an intermediate time t, the consolidating pressure $\Delta\sigma$ is partly carried by water and partly by soil, and the following relationship is obtained : $\Delta\sigma = \Delta\sigma' + \bar{u}$. The distribution of excess hydrostatic pressure $\bar{u}$ at any time t is indicated by the curve *AFB*, joining water levels, in the piezometric tubes ; this curve is known as *isochrone*, and number of such isochrones can be drawn at various time intervals t_1, t_2, t_3 etc. The slope of isochrones at any point at a given time indicates the rate of change of $\bar{u}$ with depth.

At any times t, the hydraulic head h corresponding to the excess hydrostatic pressure is given by

$$h = \frac{\bar{u}}{\gamma_w} \quad \ldots(i)$$

Hence the hydraulic gradient i is given by

$$i = \frac{\partial h}{\partial z} = \frac{1}{\gamma_w}\frac{\partial \bar{u}}{\partial z} \quad \ldots(ii)$$

Thus, the rate of change of $\bar{u}$ along the depth of the layer represents the hydraulic gradient. The velocity with which the excess pore water flows at the depth z is given by Darcy's law

$$v = ki = \frac{k}{\gamma_w}\frac{\partial \bar{u}}{\partial z} \quad \ldots(iii)$$

The rate of change of velocity along the depth of the layer is then given by

$$\frac{\partial v}{\partial z} = \frac{k}{\gamma_w}\frac{\partial^2 \bar{u}}{\partial z^2} \quad \ldots(iv)$$

Consider a small soil element of size dx, dz, and of width dy perpendicular to the xz plane. If v is the velocity of water at the entry into the elements, the velocity at the exit will be equal to $v + \frac{\partial v}{\partial z} dz$

The quantity of water entering the soil element $= v\,dxdy$

The quantity of water leaving the soil element $= \left(v + \frac{\partial v}{\partial z} dz\right) dxdy.$

Hence the net quantity of water dq squeezed out of the soil element per unit time is given by

$$\Delta q = \frac{\partial v}{\partial z}\, dx\ dy\ dz \quad \ldots(v)$$

The decrease in the volume of soil is equal to the volume of water squeezed out.

However, from Eq. 15.4. $\Delta V = -\, m_v\, V_0\, \Delta\sigma'$ ...(*vi*)

where V_0 = volume of soil element at time $t_0 = dx\ dy\ dz.$

∴ Change of volume per unit time is given by

$$\frac{\partial(\Delta V)}{\partial t} = -\, m_v\ dx\ dy\ dz\ \frac{\partial(\Delta\sigma')}{\partial t} \quad \ldots(vii)$$

Equating (*v*) and (*vii*), we get $\frac{\partial v}{\partial z} = -\, m_v \frac{\partial(\Delta\sigma')}{\partial t}$ ...(*viii*)

Now $\Delta\sigma = \Delta\sigma' + \bar{u}$, where $\Delta\sigma$ is constant.

$$\therefore \quad \frac{\partial(\Delta\sigma')}{\partial t} = -\frac{\partial \bar{u}}{\partial t} \quad \ldots(ix)$$

Hence, from (*viii*) and (*ix*), $\frac{\partial v}{\partial z} = m_v \frac{\partial \bar{u}}{\partial t}$ ...(*x*)

Combining Eqs. (*iv*) and (*x*), we get

$$\frac{\partial \bar{u}}{\partial t} = \frac{k}{m_v\, \gamma_w} \frac{\partial^2 \bar{u}}{\partial z^2} \quad \ldots(15.16)$$

or

$$\frac{\partial \bar{u}}{\partial t} = c_v \frac{\partial^2 \bar{u}}{\partial z^2} \quad \ldots(15.17)$$

where

$$c_v = \textit{coefficient of consolidation} = \frac{k}{m_v\, \gamma_w} \quad \ldots(15.18)$$

$$= \frac{k(1 + e_0)}{a_v\, \gamma_w} \quad \ldots(15.19)$$

Eq. 15.17 is the basic differential equation of consolidation which relates the rate of change of excess hydrostatic pressure to the rate of expulsion of excess pore water from a unit volume of soil during the same time interval. The term *coefficient of consolidation* c_v used in the equation is adopted to indicate the combined effects of permeability and compressibility of soil on the rate of volume change. The units of c_v are cm^2/sec.

15.6. SOLUTION OF THE CONSOLIDATION EQUATION*

The solution of the differential equation of consolidation is obtained by means of the Fourier series. The solution must satisfy the following hydraulic boundary conditions [Fig. 15.5 (*a*)] :

optimum has a steeper stress-strain curve and hence has a higher modulus of elasticity, than the one which is compacted wet of optimum (Fig. 17.12), at the same density. Soil compacted wet of optimum have *brittle failure* while soil compacted wet of optimum, and having dispersed structure, continue to increase in strength even at higher strains.

8. Shear strength : The shear strength of compacted clays depend upon (*i*) dry density, (*ii*) moulding water content, (*iii*) soil structure (*iv*) method of compaction (*v*) strain used to defined strength (*vi*) drainage condition and (*vii*) type of soil.

In general, at low strains, strength of cohesive soils compacted dry of optimum is higher than those compacted wet of optimum. Fig. 17.13 shows the failure envelope of two samples of the same soil, one compacted dry of optimum and the other compacted wet of the optimum, but both compacted at the same density. However, of higher strains, the flocculated structure of the same compacted on the dry side is broken, giving rise to ultimate strength for both the samples. The manner of compaction also influences the strength of soil sample compacted wet of optimum. It is interesting to note that the clay cores in earth dams are usually compacted wet of optimum to tolerate large settlements without cracking.

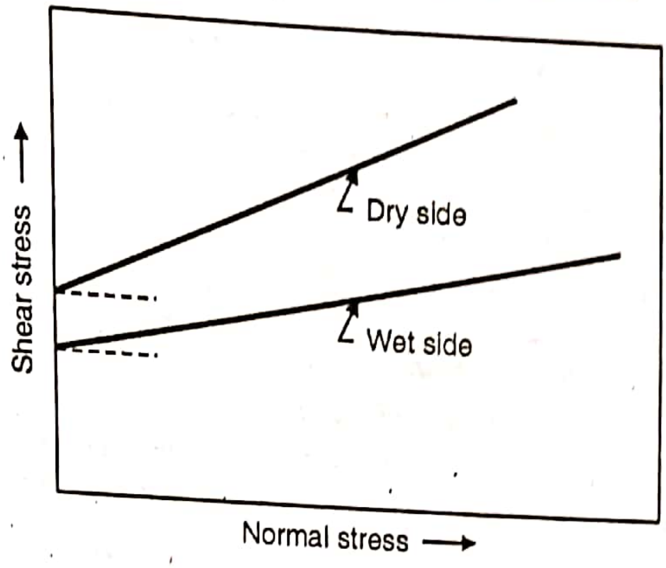

FIG. 17.13 FAILURE ENVELOPES

Example 17.1. *A laboratory compaction test on soil having specific gravity equal to 2.68 gave a maximum dry density of 1.82 g/cm³ and a water content of 17 per cent. Determine the degree of saturation, air content and percentage air voids at the maximum dry density. What would be theoretical maximum dry density corresponding to zero air voids at the optimum water content ?*

Solution.
$$\rho_d = \frac{G\rho_w}{1 + \frac{wG}{S}}$$

$$\therefore \quad 1 + \frac{0.17 \times 2.68}{S} = \frac{G\rho_w}{\rho_d} = \frac{2.68}{1.82} = 1.485$$

$$\therefore \quad S = \frac{0.17 \times 2.68}{0.485} = 0.94 = 94\% \quad ; \quad a_c = 1 - S = 1 - 0.94 = 0.06 = 6\%$$

$$\rho_d = \frac{(1 - n_a)G\rho_w}{1 + wG}$$

$$\therefore \quad (1 - n_a) = \frac{1.82(1 + 0.17 \times 2.68)}{2.68} = 0.99 \quad \text{or} \quad n_a = 1 - 0.99 = 0.01 = 1\%$$

When $n_a = 0$ $(S = 1)$, theoretical dry density at $w = 17\%$ is given by

$$\rho_d = \frac{G\rho_w}{1 + wG} = \frac{2.68 \times 1}{1 + 0.17 \times 2.68} = \mathbf{1.84\ g/cm^3}$$

The corresponding dry unit weight is

$$\gamma_d = 9.81\ \rho_d = 9.81 \times 1.84 = \mathbf{18.05\ kN/m^3}$$

Example 17.3. *Work out theoretical maximum dry density for a soil sample having sp. gr. of 2.7 and OMC = 16%. Also explain the difference in OMC value in case of Proctor test and modified proctor test for cohesive soils and granular soils.*

(Engg. Services Exam. 2001)

Solution : $\gamma_{d,\,max}$ occurs when S is maximum, i.e. when $S = 1$

$$\gamma_{d,\,max} = \frac{G\,\gamma_w}{1 + \frac{wG}{S}} = \frac{G\,\gamma_w}{1 + w\,.\,G}$$

Hence
$$\rho_{d,\,max} = \frac{G\,\rho_w}{1 + w\,G} = \frac{2.7 \times 1}{1 + 0.16 \times 2.7} = \mathbf{1.885\ g/cm^3}$$

Example 17.4. *A cohesive soil yields a maximum dry density of 1.8 g/cc at an OMC of 16% during a standard proctor test. If the values of G is 2.65, what is the degree of saturation ? What is the maximum dry density it can further compacted to?*

(Gate Exam. 1992)

Solution : Given $\rho_d = 1.8\ g/cm^3$; $w = 0.16$; $G = 2.65$

$$e = \frac{G\,\rho_w}{\rho_d} - 1 = \frac{2.65 \times 1}{1.8} - 1 = 0.4722$$

$\therefore$
$$S = \frac{w\,G}{e} = \frac{0.16 \times 2.65}{0.4722} = 0.8979 = \mathbf{89.79\%}$$

Now
$$\rho_d = \frac{G\,\rho_w}{1 + \frac{wG}{S}}\ ; \text{ when } S = 1, \text{ we get}$$

$$\rho_{d,\,max} = \frac{G\,\rho_w}{1 + w\,G} = \frac{2.65 \times 1}{1 + 0.16 \times 2.65} = \mathbf{1.861\ g/cm^3}$$

17.14. LABORATORY EXPERIMENTS

EXPERIMENT 17 : DETERMINATION OF COMPACTION PROPERTIES

Object and Scope. The object of the experiment is to determine the relationship between water content and dry density of soil using Standard Proctor Test (light compaction) or Modified Proctor Test (heavy compaction), and then to determine the optimum water content and the corresponding maximum dry density for a soil. The test also covers the determination of relationship between penetration resistance and water content for the compacted soil.

or $p_p = K_p \gamma z$ where $K_p = \cos\beta \dfrac{\cos\beta + \sqrt{\cos^2\beta - \cos^2\phi}}{\cos\beta - \sqrt{\cos^2\beta - \cos^2\phi}}$ ...(20.45)

(*b*) **Cohesive backfill** : For the case of cohesive soil, the principal stress relationship at failure is given by

$$\sigma_1 = \sigma_3 \tan^2\alpha + 2c\tan\alpha$$

For the case of passive pressure,

$$\sigma_1 = \sigma_h = p_p$$

$$\sigma_3 = \sigma_v = \gamma z$$

Substituting these values of σ_1 and σ_3, we get

$$p_p = \gamma z \tan^2\alpha + 2c\tan\alpha \quad ...(20.46)$$

or

$$p_p = \gamma z N_\phi + 2c\sqrt{N_\phi}$$

At $z = 0$, $p_p = 2c\tan\alpha$

At $z = H$, $p_p = \gamma H\tan^2\alpha + 2c\tan\alpha$

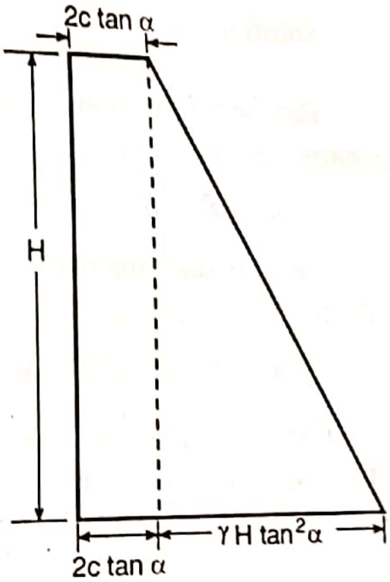

FIG. 20.13 VARIATION OF PASSIVE PRESSURE : COHESIVE BACKFILL.

Fig. 20.13 shows the pressure distribution diagram. The total pressure is given by

$$P_p = \int_0^H p_p\,dz = \frac{1}{2}\gamma H^2\tan^2\alpha + 2cH\tan\alpha \quad ...(20.47)$$

$$= \frac{1}{2}\gamma H^2 N_\phi + 2cH\sqrt{N_\phi} \quad ...(20.47a)$$

Example 20.2. *Compute the intensities of active and passive earth pressure at depth of 8 metres in dry cohesionless sand with an angle of internal friction of 30° and unit weight of 18 kN/m³. What will be the intensities of active and passive earth pressure if the water level rises to the ground level ? Take saturated unit weight of sand as 22 kN/m³.*

Solution. (*a*) ***Dry soil*** :

$$K_a = \frac{1-\sin\phi}{1+\sin\phi} = \frac{1-\sin 30°}{1+\sin 30°} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3} \quad ; \quad K_p = \frac{1+\sin\phi}{1-\sin\phi} = \frac{1}{K_a} = 3$$

$\therefore$

$$p_a = K_a\gamma H = \frac{1}{3}\times 18\times 8 = 48\ \text{kN/m}^2$$

$$p_p = K_p\gamma H = 3\times 18\times 8 = 432\ \text{kN/m}^2$$

(*b*) ***Submerged backfill*** :

$$\gamma' = \gamma_{sat} - \gamma_w = 22 - 9.81 = 12.19\ \text{kN/m}^3$$

$$p_a = K_a\gamma' H + \gamma_w H = \frac{1}{3}\times 12.19\times 8 + 9.81\times 8 \approx \mathbf{111}\ \text{kN/m}^2$$

$$p_p = K_p\gamma' H + \gamma_w H = (3\times 12.19\times 8) + (9.81\times 8) = \mathbf{371}\ \text{kN/m}^2$$

Example 20.3. *A retaining wall 4 m high, has a smooth vertical back. The backfill has a horizontal surface in level with the top of the wall. There is uniformly distributed surcharge load of 36 kN/m² intensity over the backfill. The unit weight of the backfill*

$$F_C = \frac{H_C}{H} \qquad ...(23.13)$$

Thus the factor of safety F_c with respect to cohesion, also represents the factor of safety with respect to height. It is based on the assumption that the frictional resistance of the soil is fully developed. However, the true *factor of safety* is different from F_c, and is equally applied to both cohesion and frictional resistance of soil.

Submerged slope : If the slope is submerged, γ should be replaced by γ' ; also c and φ should be determined corresponding to submerged condition.

Thus,
$$F = \frac{c + \gamma' z \cos^2 i \tan\varphi}{\gamma' z \cos i \sin i} \qquad ...(23.7\ a)$$

$$H_C = \frac{c}{\gamma'} \frac{1}{(\tan i - \tan\varphi)\cos^2 i} = \frac{c}{\gamma'} \cdot \frac{\sec^2 i}{\tan i - \tan\varphi} \qquad ...(23.8a)$$

Steady seepage along the slope : As in the case of non-cohesive soil, τ should be taken with respect to saturated weight (γ_{sat}) while σ should be computed with respect to the submerged weight. Hence Eqs. 23.7 is modified as under

$$F = \frac{c + \gamma' z \,.\, \cos^2 i \tan\varphi}{\gamma_{sat}\, z \cos i \sin i} = \frac{c}{\gamma_{sat} \cos i \sin i} + \frac{\gamma'}{\gamma_{sat}} \cdot \frac{\tan\varphi}{\tan i}$$

For the critical height $z = H_c$ corresponding to $F = 1$, we have from above

$$\gamma_{sat}\ H_C \cos i \sin i = c + \gamma' H_C \cos^2 i \tan\varphi$$

which gives
$$H_C = \frac{c}{(\gamma_{sat} \tan i - \gamma' \tan\varphi)\cos^2 i} = \frac{c}{\gamma_{sat}\left\{\tan i - \frac{\gamma'}{\gamma_{sat}} \tan\varphi\right\}\cos^2 i} \qquad ...(23.8\ b)$$

Comparing it with Eq. 23.8, it is seen that the effect of shearing resistance φ is reduced as compared to Eq. 23.8.

Example 23.1. *A long natural slope of cohesionless soil is inclined at 12° to the horizontal. Taking $\varphi = 30°$, determine the factor of safety of the slope. If the slope is completely submerged, what will be change in the factor of safety ?*

Solution : From Eq. 23.6,
$$F = \frac{\tau_C}{\tau} = \frac{\tan\varphi}{\tan i}$$

Here $\varphi = 30°,\ i = 12°$ $\qquad \therefore \quad F = \frac{\tan 30°}{\tan 12°} = \mathbf{2.72}$

Effect of submergence: When the slope is submerged, γ is replaced by γ'

$$\therefore \quad F = \frac{\tau_C}{\tau} = \frac{(\gamma' z \cos^2 i)\tan\varphi}{\gamma' z \sin i \cos i} = \frac{\tan\varphi}{\tan i} = \frac{\tan 30°}{\tan 12°} = 2.72$$

Thus, the F.S. will remain the same, except that φ is to be determined under submerged condition.

Example 23.2. *A long natural slope of sandy soil ($\varphi = 25°$) is inclined at 10° to the horizontal. The water table is at the surface and the seepage is parallel to the slope.*

If the saturated unit weight of the soil is 19.5 kN/m³, determine the factor of safety of the slope.

Solution : From Eq. 23.6 (*a*), $F = \dfrac{\gamma' \tan \varphi}{\gamma_{sat} \tan i} = \dfrac{(19.5 - 9.81) \tan 25°}{19.5 \tan 10°} = \mathbf{1.31}$

Example 23.3. *A long natural slope in a c – φ soil is inclined at 12° to the horizontal. The water table is at the surface and the seepage is parallel to the slope. If a plane slip has developed at a depth of 4 m, determine the factor of safety.*

Take $c = 8\ kN/m^2$, $\varphi = 22°$ *and* $\gamma_{sat} = 19\ kN/m^3$.

Solution

From Eq. 23.7 (*b*) $F = \dfrac{c + \gamma' z \cos^2 i \tan \varphi}{\gamma_{sat}\, z \cos i \sin i} = \dfrac{8 + (19 - 9.81) \times 4 \cos^2 12° \tan 22°}{19 \times 4 \cos 12° \sin 12°} = \mathbf{1.44}$