The product rule, quotient rule and the function of a function rule continues to hold good here also.

Observe the following comparisons.

| Ordinary derivative | Partial derivatives |
|---|---|
| 1. $y = 3x^2 + 6x + 7$<br>$\frac{dy}{dx} = 6x + 6 + 0 = 6x + 6$ | 1. $u = 3x^2 y + 6xy^2 + 7$<br>$\frac{\partial u}{\partial x} = (6x)y + 6 \cdot 1 \cdot y^2 + 0 = 6xy + 6y^2$<br>[$y$ is treated as constant]<br>$\frac{\partial u}{\partial y} = 3x^2 \cdot 1 + 6x \cdot 2y + 0 = 3x^2 + 12xy$<br>[$x$ is treated as constant] |
| 2. $y = e^{4x+3}$<br>$\frac{dy}{dx} = e^{4x+3} \cdot \frac{d}{dx}(4x+3)$<br>$= e^{4x+3} \cdot 4 = 4e^{4x+3}$ | 2. $u = e^{4x+3y}$<br>$\frac{\partial u}{\partial x} = e^{4x+3y} \cdot \frac{\partial}{\partial x}(4x+3y)$<br>$= e^{4x+3y} \cdot (4+0) = 4e^{4x+3y}$<br>$\frac{\partial u}{\partial y} = e^{4x+3y} \cdot \frac{\partial}{\partial y}(4x+3y)$<br>$= e^{4x+3y} \cdot (0+3) = 3e^{4x+3y}$ |
| 3. $y = \sin 5x$<br>$\frac{dy}{dx} = \cos 5x \cdot 5 = 5 \cos 5x$ | 3. $u = \sin(xy)$<br>$\frac{\partial u}{\partial x} = \cos(xy) \frac{\partial}{\partial x}(xy) = \cos(xy) \cdot y$<br>$\frac{\partial u}{\partial y} = \cos(xy) \frac{\partial}{\partial y}(xy) = \cos(xy) \cdot x$ |
| 4. $y = \tan^{-1}(2/x)$<br>$\frac{dy}{dx} = \frac{1}{1+(2/x)^2} \cdot \frac{d}{dx}\left(\frac{2}{x}\right)$<br>$= \frac{x^2}{x^2+4} \cdot -\frac{2}{x^2} = -\frac{2}{x^2+4}$ | 4. $u = \tan^{-1}(y/x)$, $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$<br>*ie.*, $= \frac{x^2}{x^2+y^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$<br>$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$<br>$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$ |
| 5. $\frac{d}{dx}[f(y)] = f'(y) \frac{dy}{dx}$ | 5. If $r$ is a function of $x$ and $y$,<br>$\frac{\partial}{\partial x}[f(r)] = f'(r) \frac{\partial r}{\partial x}$<br>$\frac{\partial}{\partial y}[f(r)] = f'(r) \frac{\partial r}{\partial y}$ |

## General principle of partial differentiation

Given a function of many independent variables, the derivative of this function with respect to a particular independent variable, keeping (*treating*) all other independent variables as constants is the general principle of partial differentiation.

## Higher order partial derivatives

These are also analogous with the higher order ordinary derivatives.

Let us suppose that $u = f(x, y)$. The development of higher order partial derivatives is as exhibited below.

$$u = f(x, y)$$

**First order partial derivatives**

$$u_x = \frac{\partial u}{\partial x} \qquad u_y = \frac{\partial u}{\partial y}$$

**Second order partial derivatives**

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \qquad u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x} \longleftarrow \text{Mixed partial derivatives} \longrightarrow u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y}$$

⟶ and so-on ⟵

It is very important to note that;

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y} \quad \text{or} \quad u_{yx} = u_{xy}$$

**WORKED PROBLEMS**

**Set – 3**

## Type - 1 Direct partial derivatives

Given an explicit function of more than one independent variable we find the required partial derivatives just keeping in mind the *general principle of partial differentiation* stated earlier along with the well acquainted rules of differentiation. The following note on symmetric functions will be highly useful for certain problems.

**Note : Symmetric function** : *A function $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x)$ and a function $f(x, y, z)$ is said to be symmetric if $f(x, y, z) = f(y, z, x) = f(z, x, y)$. In general we can say that a function of several variables is symmetric if the function remains unchanged (invariant) when the variables are cyclically rotated. Observe the following examples.*

(i) $x + y,\ x^2 + y^2,\ \dfrac{x^2 + y^2}{x + y},\ x^2 + xy + y^2,\ \log\sqrt{x^2 + y^2}$ *etc.*

*are symmetric functions of two variables as it can be easily seen that when $x$ is replaced by $y$ and $y$ by $x$ the functions remain the same.*

(ii) $x^2+y^2+z^2,\ xy+yz+zx,\ x/y+y/z+z/x,\ \log(x+y+z),$

$x^3+y^3+z^3-3xyz$ etc.

*are symmetric functions of three variables.*

***It is very important to note that,*** *if we have a symmetric function of three variables say $u = f(x, y, z)$ then by just computing $u_x$ or $u_{xx}$ or $u_{xy}$ we can simply writedown easily the other partial derivatives $(u_y, u_z)$ or $(u_{yy}, u_{zz})$ or $(u_{yz}, u_{zx})$ by simple guess work. There is no need to show the working of similar computation of partial derivatives*

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**69.** *If* $u = x^3-3xy^2+x+e^x\cos y+1$, *show that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

We have $u = x^3-3xy^2+x+e^x\cos y+1$

$\therefore \quad \dfrac{\partial u}{\partial x} = 3x^2-3y^2+1+e^x\cos y$ ... ($y$ *is treated as constant.*)

Differentiating this *w.r.t* $x$ partially again,

$\dfrac{\partial^2 u}{\partial x^2} = 6x+e^x\cos y$ ... (*Again* $y$ *is treated as constant.*)

Next, $\dfrac{\partial u}{\partial y} = -6xy-e^x\sin y$ ... ($x$ *is treated as constant.*)

Differentiating this *w.r.t* $y$ partially again,

$\dfrac{\partial^2 u}{\partial y^2} = -6x-e^x\cos y$ ... (*Again* $x$ *is treated as constant.*)

Now $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = 6x+e^x\cos y-6x-e^x\cos y = 0$

**Thus we have proved the desired result.**

---

**70.** *If* $u = e^{-2\pi^2 t}\sin\pi x\sin\pi y$ *show that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial u}{\partial t}$

>> We have $u = e^{-2\pi^2 t}\sin\pi x\sin\pi y$

$\therefore \quad \dfrac{\partial u}{\partial x} = e^{-2\pi^2 t}(\pi\cos\pi x)\sin\pi y$ ... ($t$ & $y$ *are treated as constants*)

Differentiating this *w.r.t* $x$ partially again,

$$\frac{\partial^2 u}{\partial x^2} = e^{-2\pi^2 t}(-\pi^2 \sin \pi x) \sin \pi y = -\pi^2 u$$

$$\frac{\partial u}{\partial y} = e^{-2\pi^2 t} \sin \pi x (\pi \cos \pi y) \quad \ldots (t \ \& \ x \text{ are treated as constants})$$

Differentiating this *w.r.t* $y$ partially again,

$$\frac{\partial^2 u}{\partial y^2} = e^{-2\pi^2 t} \sin \pi x (-\pi^2 \sin \pi y) = -\pi^2 u$$

Hence LHS $= \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -\pi^2 u - \pi^2 u = -2\pi^2 u$ ...(1)

Also $\dfrac{\partial u}{\partial t} = e^{-2\pi^2 t}(-2\pi^2) \sin \pi x \sin \pi y$ ... ($x$ & $y$ *are treated as constants*)

*ie.,* RHS $= \dfrac{\partial u}{\partial t} = -2\pi^2 u$ ...(2)

Thus from (1) and (2) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial t}$

---

**71.** *If* $u = \log\left(\dfrac{x^2+y^2}{x+y}\right)$ *show that* $x u_x + y u_y = 1$

>> $u = \log(x^2+y^2) - \log(x+y)$

$\therefore$ $u_x = \dfrac{1}{x^2+y^2} \cdot 2x - \dfrac{1}{x+y} \cdot 1$

and $u_y = \dfrac{1}{x^2+y^2} \cdot 2y - \dfrac{1}{x+y} \cdot 1$

Now, $x u_x + y u_y = \dfrac{2x^2}{x^2+y^2} - \dfrac{x}{x+y} + \dfrac{2y^2}{x^2+y^2} - \dfrac{y}{x+y}$

$$= \frac{2(x^2+y^2)}{x^2+y^2} - \frac{(x+y)}{x+y} = 2 - 1 = 1$$

Thus $x u_x + y u_y = 1$

80. *If* $u = e^{ax+by} f(ax - by)$, *prove that* $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$

\>> $u = e^{ax+by} f(ax-by)$, by data.

$$\frac{\partial u}{\partial x} = e^{ax+by} \cdot f'(ax-by)\,a + a\,e^{ax+by} f(ax-by)$$

or $$\frac{\partial u}{\partial x} = a\,e^{ax+by} f'(ax-by) + a\,u$$

Next, $$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by) \cdot (-b) + b\,e^{ax+by} f(ax-by)$$

or $$\frac{\partial u}{\partial y} = -b\,e^{ax+by} f'(ax-by) + bu$$

Now consider LHS $= b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y}$ by using (1) and (2).

$$= b\left\{ae^{ax+by} f'(ax-by) + au\right\} + a\left\{-be^{ax+by} f'(ax-by) + bu\right\}$$

$$= abe^{ax+by} f'(ax-by) + abu - abe^{ax+by} f'(ax-by) + abu$$

$$= 2\,abu = \text{RHS}$$

Thus $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2\,abu.$

## Problems on symmetric functions

83. *If* $u = \log\sqrt{x^2+y^2+z^2}$, *show that* $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 1$

\>> By data $u = \log\sqrt{x^2+y^2+z^2} = \frac{1}{2}\log(x^2+y^2+z^2)$

The given $u$ is a symmetric function of $x, y, z,$

*(It is enough if we compute only one of the required partial derivative)*

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

*ie.,* $$= \frac{(x^2+y^2+z^2)\,1 - x\cdot 2x}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} \qquad \ldots(1)$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2} \qquad \ldots(2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \quad \ldots(3)$$

Adding (1), (2), and (3) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Thus $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 1$

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**84.** *If* $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ *then show that* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$

>> $u = (x^2+y^2+z^2)^{-1/2}$ is a symmetric function of $x, y, z$,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x = -(x^2+y^2+z^2)^{-3/2}\cdot x$$

$$\therefore \quad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\left\{(x^2+y^2+z^2)^{-3/2}\cdot x\right\}$$

$$= -\left\{(x^2+y^2+z^2)^{-3/2}\cdot 1 + x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2x\right\}$$

$$= -\left\{(x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2}\right\}$$

*ie.,* $$\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \ldots(1)$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \ldots(2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \ldots(3)$$

Adding the results (1), (2) and (3) we have,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$$

Thus $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$

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85. *If* $u = \log(x^3+y^3+z^3-3xyz)$ *then prove that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

*and hence show that* $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

>> $u = \log(x^3+y^3+z^3-3xyz)$ is a symmetric function.

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz} \quad \ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad \ldots (3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x^3+y^3+z^3-3xyz)}$$

Recalling a standard elementary result,

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

we have,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

Thus $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

Further $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ by using the earlier result.}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

Thus $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

## 2.33 Homogeneous function and Euler's theorem

**Definition :** A function $u = f(x, y)$ is said to be a *homogeneous function of degree* $n$ if it can be expressed in the form $x^n g(y/x)$ or $y^n g(x/y)$, $g$ being any arbitrary function.

Similarly a function $u = f(x, y, z)$ is said to be a *homogeneous function of degree* $n$ if it can be expressed in the form $x^n g(y/x, z/x)$ or $y^n g(x/y, z/y)$ or $z^n g(x/z, y/z)$.

Observe the following examples.

1. $u = 3x + 4y$ ; $u = x[3 + 4(y/x)] = x^1 g(y/x)$

$\Rightarrow$ $u$ is homogeneous of degree 1 ( *ie.*, $n = 1$ )

2. $u = x^2 \sin(y/x) + y^2 \cos(y/x) + xy$

$$u = x^2 \left[\sin(y/x) + (y/x)^2 \cos(y/x) + (y/x)\right] = x^2 g(y/x)$$

$\Rightarrow$ $u$ is homogeneous of degree 2 ( *ie.*, $n = 2$ )

3. $u = x^2 y + x y^2$ *ie.*, $u = x^3\left[(y/x) + (y/x)^2\right] = x^3 g(y/x)$

$\Rightarrow$ $u$ is homogeneous of degree 3 ( *ie.*, $n = 3$ )

4. $u = x^3 y \tan^{-1}(x/y) + x y^3 \sec^{-1}(x/y)$

$$u = y^4\left[(x/y)^3 \tan^{-1}(x/y) + (x/y)\sec^{-1}(x/y)\right] = y^4 g(x/y)$$

$\Rightarrow$ $u$ is homogeneous of degree 4 ( *ie.*, $n = 4$ )

5. $$u = \frac{x^2\sqrt{y} + y^2\sqrt{x}}{\sqrt{x} - \sqrt{y}}$$

*ie.*, $$u = \frac{x^{5/2}\left[\sqrt{y/x} + (y/x)^2\right]}{\sqrt{x}\left[1 - \sqrt{y/x}\right]}$$

*ie.*, $$u = x^2\left[\frac{\sqrt{y/x} + (y/x)^2}{1 - \sqrt{y/x}}\right] = x^2 g(y/x)$$

$\Rightarrow$ $u$ is homogeneous of degree 2 ( *ie.*, $n = 2$ )

6. $$u = \frac{2x^2 + 3y^2 + 4z^2}{\sqrt{x} + \sqrt{y} + \sqrt{z}}$$

*ie.*, $$u = \frac{x^2[2 + 3(y/x)^2 + 4(z/x)^2]}{\sqrt{x}[1 + \sqrt{y/x} + (\sqrt{z}/x)]}$$

ie., $$u = x^{3/2}\left[\frac{2+3(y/x)^2+4(z/x)^2}{1+\sqrt{y/x}+\sqrt{z/x}}\right] = x^{3/2}\, g(y/x,\ z/x)$$

$\Rightarrow$ $u$ is homogeneous of degree $3/2$ ( ie., $n = 3/2$ )

**Euler's theorem on homogeneous functions**

**Statement :** *If* $u = f(x, y)$ *is a homogeneous function of degree* $n$ *then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$$

**Proof :** Since $u = f(x, y)$ is a homogeneous function of degree $n$ we have by the definition,

$$u = x^n g(y/x) \qquad \ldots (1)$$

Let us differentiate this *w.r.t* $x$ and also *w.r.t* $y$,

$$\therefore \quad \frac{\partial u}{\partial x} = x^n \cdot g'(y/x)\cdot\left(-\frac{y}{x^2}\right) + nx^{n-1} g(y/x)$$

ie., $$\frac{\partial u}{\partial x} = -x^{n-2}\, y\, g'(y/x) + n\,x^{n-1} g(y/x) \qquad \ldots (2)$$

Also $$\frac{\partial u}{\partial y} = x^n \cdot g'(y/x)\cdot\left(\frac{1}{x}\right)$$

ie., $$\frac{\partial u}{\partial y} = x^{n-1} g'(y/x) \qquad \ldots (3)$$

Now consider $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ as a consequence of (2) & (3).

$$= x\left[-x^{n-2}\, y\, g'(y/x) + n\,x^{n-1} g(y/x)\right] + y\left[x^{n-1} g'(y/x)\right]$$

$$= -x^{n-1}\, y\, g'(y/x) + n\,x^n g(y/x) + x^{n-1}\, y\, g'(y/x)$$

$$= n\cdot x^n g(y/x)$$

$= n\,u$, by using (1)

Thus we have proved **Euler's theorem :**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u \;;\quad x\,u_x + y\,u_y = n\,u$$

**Note-1 :** *This theorem can be extended to more than two independent variables also. Suppose* $u = f(x, y, z)$ *is a homogeneous function of degree* $n$*, then the associated Euler's theorem is given by*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\,u$$

**Note - 2 :** *The theorem can also be extended for second order partial derivatives and we prove the associated result.*

**Statement :** *If* $u = f(x, y)$ *is a homogeneous function of degree* $n$ *then*

$$\boxed{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u}$$

**Proof :** Since $u = f(x, y)$ is a homogeneous function of degree $n$, we have Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \qquad \ldots (1)$$

Differentiating (1) partially *w.r.t* $x$ and also *w.r.t* $y$ we get,

$$\left( x \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} \right) + y \frac{\partial^2 u}{\partial x \, \partial y} = n \frac{\partial u}{\partial x}$$

Also, $$x \frac{\partial^2 u}{\partial y \, \partial x} + \left( y \frac{\partial^2 u}{\partial y^2} + 1 \cdot \frac{\partial u}{\partial y} \right) = n \frac{\partial u}{\partial y}$$

We shall now multiply (2) by $x$ and (3) by $y$.

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \, \partial y} = nx \frac{\partial u}{\partial x} \text{ and}$$

$$xy \frac{\partial^2 u}{\partial y \, \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = ny \frac{\partial u}{\partial y}$$

Adding these using the fact that $\frac{\partial^2 u}{\partial y \, \partial x} = \frac{\partial^2 u}{\partial x \, \partial y}$ we get,

$$\left( x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right) + \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = n \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

ie., $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + nu = n(nu), \text{ by using (1).}$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(nu) - nu = n(n-1)u$$

Thus $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

ie., $$\boxed{x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n(n-1)u}$$

Note :

(1) This result can also be put in the form

$$\left( x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)^2 u = n(n-1)u$$

(2) The result can also be established by starting from the basic definition of homogeneous function.

## WORKED PROBLEMS
### Set – 5

***Verify Euler's theorem for the following functions.***

95. $u = x^3 + x^2 y + x y^2 + y^3$

96. $u = y^n \log(x/y)$

95. $u = x^3 + x^2 y + x y^2 + y^3$

**Note :** *We have to first ensure that the given function is homogeneous of some degree and then compute the first order partical derivatives of the given function, which will enable us to verify the theorem.*

With reference to the given $u$ we have,

$$u = x^3 \left[ 1 + (y/x) + (y/x)^2 + (y/x)^3 \right] = x^3 g(y/x)$$

$\Rightarrow$ $u$ is homogeneous of degree 3 $\therefore$ $n = 3$

Also form the given $u$ we have,

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy + y^2, \quad \frac{\partial u}{\partial y} = x^2 + 2xy + 3y^2$$

We have Euler's theorem : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$$\text{LHS} = x(3x^2 + 2xy + y^2) + y(x^2 + 2xy + 3y^2)$$

$$= 3x^3 + 2x^2 y + x y^2 + x^2 y + 2 x y^2 + 3y^3$$

$$= 3x^3 + 3x^2 y + 3 x y^2 + 3y^3$$

$$= 3(x^3 + x^2 y + x y^2 + y^3) = 3u = \text{RHS} \quad \text{since } n = 3$$

**Thus Euler's theorem is verified.**

**96.** $u = y^n \log(x/y)$

This is a homogeneous function of degree $n$ by the definition.

$$\therefore \quad \frac{\partial u}{\partial x} = y^n \cdot \frac{1}{(x/y)} \cdot \frac{1}{y} = \frac{y^n}{x} \qquad \ldots (1)$$

$$\frac{\partial u}{\partial y} = y^n \cdot \frac{1}{(x/y)} \cdot \left(-\frac{x}{y^2}\right) + n\,y^{n-1} \log(x/y)$$

$$ie., \quad \frac{\partial u}{\partial y} = -y^{n-1} + n\,y^{n-1} \log(x/y) \qquad \ldots (2)$$

From (1) and (2) we have,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x \cdot \frac{y^n}{x} + y\left[-y^{n-1} + n\,y^{n-1}\log(x/y)\right]$$

$$= y^n - y^n + n\,y^n \log(x/y)$$

$$= n\,[\,y^n \log(x/y)\,] = n\,u$$

**Thus Euler's theorem is verified.**

## Standard type of problems by applying Euler's theorem

Given $u = f(x, y)$ or $u = f(x, y, z)$ or their equivalents (*like* $\log u$, $\sin u$,) as homogeneous functions the computation of

(i) $x\,u_x + y\,u_y$ (ii) $x^2 u_{xx} + 2\,x\,y\,u_{xy} + y^2 u_{yy}$

can be done very easily by applying Euler's theorem.

## Flow chart for solving problems

We have to first express $u$ or its equivalent in the appropriate form according to the definition of homogeneous function so that $n$ will be known.

↓

We then apply Euler's theorem to $u$ or to its equivalent to obtain the result (i)

↓

If the given $u$ itself is homogeneous we can as well directly apply the extension form of Euler's theorem to obtain the result (ii)

↓

However if an equivalent form of $u$ is homogeneous it will be easier to obtain the result (ii) starting from the obtained result (i) proceeding on the same lines as we derived the extension of Euler's theorem involving second order partial derivatives.

**97.** *If* $u = \dfrac{x^3+y^3}{\sqrt{x+y}}$ *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{5}{2}u$

We have $u = \dfrac{x^3+y^3}{\sqrt{x+y}}$

*ie.,* $u = \dfrac{x^3[1+(y/x)^3]}{\sqrt{x}[\sqrt{1+(y/x)}]} = x^{5/2}\left\{\dfrac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\} = x^{5/2}g(y/x)$

$\Rightarrow$ $u$ is homogeneous of degree $5/2$ $\therefore$ $n = 5/2$

We have Euler's theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

Putting $n = 5/2$, we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{5}{2}u$

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**98.** *If* $u = \sqrt{x^4+y^4}\tan^{-1}(y/x)$, *then show that*

$$x u_x + y u_y = 2u = x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$$

>> $u = \sqrt{x^4+y^4}\tan^{-1}(y/x)$ by data.

$= \sqrt{x^4[1+(y/x)^4]}\ \tan^{-1}(y/x)$

$= x^2\left\{\sqrt{1+(y/x)^4}\tan^{-1}(y/x)\right\} = x^2 g(y/x)$

$\Rightarrow$ $u$ is homogeneous of degree 2 $\therefore$ $n = 2$

We have Euler's theorem and its extension given by

$$x u_x + y u_y = nu \quad \ldots(1)$$

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n(n-1)u \quad \ldots(2)$$

Puting $n = 2$ in (1) and (2) we obtain

$x u_x + y u_y = 2u$ and $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 2(2-1)u = 2u$

**Thus we have proved the required result.**

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## 2.34 Total differentiation

If $u = f(x, y)$ then the total differential or the exact differential of $u$ is defined as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \ldots (1)$$

## Differentiation of composite and implicit functions

If $u = f(x, y)$ where $x$ and $y$ are functions of the independent variable $t$ then *$u$ is said to be a composite function of the single variable $t$.*

Also if $u = f(x, y)$ where both $x$ and $y$ are functions of two independent variables $r, s$ then $u$ *is said to be a composite function of the two variables $r$ and $s$.*

The principle of differentiation of composite function is very much similar to that of the function of a function rule associated with the ordinary derivative of a function of a single independent variable.

We discuss two types involving partial derivatives.

Type - (i). Total derivative rule

If $u = f(x, y)$ where $x = x(t)$ and $y = y(t)$ then $u$ is a composite function of the single variable $t$. Therefore in principle we should be able to differentiate $u$ with respect to $t$ which is an ordinary derivative.

Thus we have with reference to (1),

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \ldots (2)$$

*This is called as the total derivative of $u$.*

Type - (ii). Chain rule

If $u = f(x, y)$ where $x = x(r, s)$ and $y = y(r, s)$ then $u$ is a composite function of two independent variables $r, s$. Therefore in principle we should we able to differentiate $u$ *w.r.t* $r$ and also *w.r.t* $s$ partially. Thus we have the following chain rules for the two partial derivatives. It is convenient to write the rule having the data analysed in the following format.

$$u \to (x, y) \to (r, s) \Rightarrow u \to (r, s) \begin{cases} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial s} \end{cases}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \; ; \; \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \ldots (3)$$

**Note - 1.** *The rules (2) and (3) can be established from the basic limit form definition of a partial derivative.*

**2.** *The rules (2) and (3) can be extended to functions involving more than two independent variables.*

**3.** *The rules (2) and (3) can be successively applied for getting higher order derivatives of the given function.*

**4.** *The symbol* $\rightarrow$ *is used only to indicate the composition of the variables so that the associated rule can be written conveniently.*

## WORKED PROBLEMS
### Set – 5

**Flow chart for solving problems (Total derivative & Chain rule)**

| We have to analyse the composition of the variables and write the appropriate formula. |
|---|

↓

| We then substitute for the possible derivatives in the formula and simplify according to the requirement of the desired result. |
|---|

***Find the total differential of the following functions.***

**111.** $u = x^3 + xy^2 + x^2y + y^3$ **112.** $x = r\sin\theta\cos\phi$

**111.** We have $u = x^3 + xy^2 + x^2y + y^3$ ; $u \rightarrow (x, y)$

$$\therefore \quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus, $du = (3x^2 + y^2 + 2xy)\,dx + (2xy + x^2 + 3y^2)\,dy$

**112.** We have $x = r\sin\theta\cos\phi$ ; $x \rightarrow (r, \theta, \phi)$

$$\therefore \quad dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi$$

Thus $dx = (\sin\theta\cos\phi)\,dr + (r\cos\theta\cos\phi)\,d\theta + (-r\sin\theta\sin\phi)\,d\phi$

113. $z = x y^2 + x^2 y$ ; $x = at, \; y = 2at$

$|z \to (x, y) \to t| \;\Rightarrow\; z \to t$ & $\dfrac{dz}{dt}$ is the total derivative.

$$\therefore \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)\,a + (2xy + x^2)\,2a$$

$$= (4a^2t^2 + 4a^2t^2)\,a + (4a^2t^2 + a^2t^2)\,2a$$

$$= (8a^2t^2)\,a + (5a^2t^2)\,2a = 8a^3t^2 + 10a^3t^2 = 18a^3t^2$$

$\therefore$ the total derivative $\dfrac{dz}{dt} = 18a^3t^2$ . . . (1)

Now by direct substitution we have,

$$z = xy^2 + x^2y = (at)(2at)^2 + (at)^2(2at) = 4a^3t^3 + 2a^3t^3 = 6a^3t^3$$

*ie.,* $z = 6a^3t^3$ and differentiating *w.r.t* $t$,

$\dfrac{dz}{dt} = 6a^3 \cdot 3t^2 = 18a^3t^2$ . . . (2

Thus from (1) & (2) the **result is verified.**

**123.** *If* $z = f(x, y)$ *where* $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$ *prove that* $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$

>> $\left\{ z \to (x, y) \to (u, v) \right\} \Rightarrow z \to (u, v)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

*ie.,*
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y}(-e^{-u}) \qquad \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \qquad \ldots (2)$$

Consider, RHS $= \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$ and $(1) - (2)$ yields

$$\frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{v}) = \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

$$\therefore \quad \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

**Thus RHS = LHS**