UNIVERSITY OF AGRICULTURAL SCIENCES, GKVK, BANGALORE
ENGINEERING MATHEMATICS – MAT111 (2+1)

INTEGRAL CALCULUS

Double and Triple integral:

Consider double integral of the form $I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)dx\,dy$

Case 1. Let x_1, x_2 and y_1, y_2 be constants then I can be evaluated in any order.

Case 2. Let x_1, x_2 be constants and y_1, y_2 are functions of x then first integrate w.r.t 'y' then w.r.t. 'x'.

$i.e.\ I = \int_{x_1}^{x_2} \left\{ \int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy \right\} dx$

Case 3. Let y_1, y_2 be constants and x_1, x_2 are functions of y then first integrate w.r.t 'x' then w.r.t 'y'. $\quad i.e\ \ I = \int_{y_1}^{y_2} \left\{ \int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx \right\} dy$

Problems:

1. Evaluate $\int_0^1 \int_0^6 xy\,dx\,dy$

Sol. Let $I = \int_0^1 \int_0^6 xy\,dx\,dy$, integrate w.r.t y, we get

$\Rightarrow I = \int_{x=0}^1 x\left[\frac{y^2}{2}\right]_0^6 dx \Rightarrow I = 18\int_0^1 x\,dx = (18)\left[\frac{x^2}{2}\right]_0^1 = 9.$

2. Evaluate $\int_0^b \int_0^a (x^2+y^2)\,dx\,dy$

Sol. Let $= \int_{x=0}^b \int_{y=0}^a (x^2+y^2)\,dx\,dy$, integrate w.r.t y, we get

$I = \int_{x=0}^b \left[x^2y + \frac{y^3}{3}\right]_0^a dx = \int_{x=0}^b \left[x^2a + \frac{a^3}{3}\right] dx$

$I = \left[\frac{x^3}{3}a + \frac{a^3}{3}x\right]_0^b = \frac{ab^3}{3} + \frac{a^3b}{3} = \frac{ab(a^2+b^2)}{3}$

3. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,dx\,dy$ **[V.T.U-2004]**

Sol. Let $I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} xy\,dx\,dy$, integrating w.r.t 'y' we get

$I = \int_{x=0}^1 x\left[\frac{y^2}{2}\right]_{y=x}^{\sqrt{x}} dx = \int_{x=0}^1 \frac{x}{2}\left[\left(\sqrt{x}\right)^2 - x^2\right] dx = \frac{1}{2}\int_{x=0}^1 (x^2 - x^3)dx$

$I = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = \frac{1}{24}$

4. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$ **[V.T.U-2000]**

Sol. Let $I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dx\,dy$, integrate w.r.t y, we get

$I = \int_{x=0}^1 \left[x^2y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx = \int_{x=0}^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right] dx$

$I = \int_{x=0}^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - 4\frac{x^3}{3}\right] dx = \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{1}{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4}{3}\frac{x^4}{4}\right]_0^1 = \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{3}{35}$

5. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3y\,dx\,dy$ **[V.T.U-2006, Jan-2017]**

Sol. Let $I = \int_{y=0}^1 \int_{x=0}^{\sqrt{1-y^2}} x^3y\,dx\,dy$, integrate w.r.t x, we get

$\Rightarrow I = \int_{y=0}^{1} y\left[\frac{x^4}{4}\right]_0^{\sqrt{1-y^2}} dy \Rightarrow I = \frac{1}{4}\int_{y=0}^{1} y\,(1-y^2)^2 dy = \frac{1}{4}\int_{y=0}^{1} y\,(1+y^4-2y^2)dy$

$\Rightarrow I = \int_0^1 (y+y^5-2y^3)dy = \frac{1}{4}\left[\frac{y^2}{2}+\frac{y^6}{6}-\frac{2y^4}{4}\right]_0^1 = \frac{1}{4}\left[\frac{1}{2}+\frac{1}{6}-\frac{1}{2}\right] = \frac{1}{24}.$

6. Evaluate $\int_0^{\pi}\int_0^{a\cos\theta} r\, sin\theta\, dr\, d\theta$

Sol. Let $I = \int_0^{\pi}\int_0^{a\cos\theta} r\, sin\theta\, dr\, d\theta \Rightarrow I = \int_{\theta=0}^{\pi}\int_{r=0}^{a\cos\theta} r\, sin\theta\, dr\, d\theta$

$\Rightarrow I = \int_{\theta=0}^{\pi} sin\theta\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi} a^2\cos^2\theta.\sin\theta.d\theta$

$\Rightarrow -\frac{a^2}{2}\int_0^{\pi} t^2 dt = -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_0^{\pi}$ $(\because put\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt\ \ or\ \ \sin\theta\, d\theta = -dt)$

$\Rightarrow I = \left(-\frac{a^2}{2}\right)\left[\frac{\cos^3\theta}{3}\right]_0^{\pi} = \left(-\frac{a^2}{2}\right)\left[-\frac{1}{3}-\frac{1}{3}\right] = \left(-\frac{a^2}{2}\right)\left(-\frac{2}{3}\right) = \frac{a^2}{3}$

7. Evaluate $\int_0^1\int_0^{x^2} e^{y/x} dy\, dx$ **[June-2014]**

Sol. Let $I = \int_0^1\int_0^{x^2} e^{\frac{y}{x}} dy\, dx = \int_{x=0}^{1}\int_{y=0}^{x^2} e^{\frac{y}{x}} dy\, dx$, integrate w.r.t y we get

$\Rightarrow I = \int_{x=0}^{1}\left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx = \int_{x=0}^{1} x\left(e^{\frac{x^2}{x}} - e^0\right)dx = \int_0^1 x(e^x-1)dx$

Apply Bernoulli's rule we get

$\Rightarrow I = \left[x(e^x-x)-(1)\left(e^x-\frac{x^2}{2}\right)\right]_0^1 = \left[xe^x - x^2 - e^x + \frac{x^2}{2}\right]_0^1$

$\Rightarrow \left[\left(e-1-e+\frac{1}{2}\right)-(0-0-1+0)\right] = -\frac{1}{2}+1 = \frac{1}{2}$

Evaluation of Triple Integrals:

Consider the triple integration of the form $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dx\,dy\,dz$.

Here the method is to evaluate is similar to that of double integrals.

If $x_1,\ x_2$ are constants; y_1, y_2 are either constants or functions x and z_1, z_2 are either constants or functions of x and y then the integral is evaluated as follows:

First integrate $f(x,y,z)$ w.r.t z between the limits z_1 and z_2 assuming x and y as constants. Then integrate the resulting expression w.r.t y between the limits y_1 and y_2, Assuming x as constant. Then finally integrate the obtained result w.r.t x between the limits x_1 and .

$$i.e.\ I = \int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right] dy\right] dx$$

1. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$. **[Jae-2011, Jan-2017, 15]**

Sol. Let $I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz = \int_{z=-1}^{1}\left[\int_{x=0}^{z}\left\{\int_{y=x-z}^{x+z}(x+y+z)dy\right\}dx\right]dz$

$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z} dx\,dz$, on simplifying we get

$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left\{x[(x+z)-(x-z)]+\frac{1}{2}[(x+z)^2-(x-z)^2]+z[(x+z)-(x-z)]\right\}dxdz$

$I = \int_{z=-1}^{1}\int_{x=0}^{z}(2xz+2xz+2z^2)\,dx\,dz = \int_{z=-1}^{1}\int_{x=0}^{z}(4xz+2z^2)\,dx\,dz$

$I = \int_{z=-1}^{1}[z(2x^2)+2z^2x]_{x=0}^{z}\,dz = \int_{z=-1}^{1}(2z^3+2z^3)dz = \left[\frac{4z^4}{4}\right]_{-1}^{1} = \mathbf{0}$

2. Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dz\,dy\,dx$ **[June-2013, 14, Jan-2016]**

Sol. Let $I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a}(x^2+y^2+z^2)dz\,dy\,dx$

$I = \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=-a}^{a} dy\,dx$

$i.e\ I = \int_{x=-c}^{c}\int_{y=-b}^{b}\left(2ax^2+2ay^2+\frac{2a^3}{3}\right)dy\,dx = \int_{x=-c}^{c}\left[2ax^2y+\frac{2ay^3}{3}+\frac{2a^3}{3}y\right]_{y=-b}^{b} dx$

$\Rightarrow I = \int_{x=-c}^{c}\left(4abx^2+\frac{4ab^3}{3}+\frac{4a^3b}{3}\right)dx = \left[\frac{4abx^3}{3}+\frac{4ab^3}{3}x+\frac{4a^3b}{3}x\right]_{-c}^{c}$

$\Rightarrow I = \frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8a^3bc}{3} = \frac{8abc(a^2+b^2+c^2)}{3}$

3. Evaluate $\int_{0}^{a}\int_{0}^{x}\int_{0}^{x+y} e^{x+y+z}dx\,dy\,dz$. **[June-2012, Jan-2017, 13]**

Sol. Let $I = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y+z}dz\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x}.e^{y}.e^{z}dz\,dy\,dx$.

$\Rightarrow I = \int_{x=0}^{a}\int_{y=0}^{x} e^{x}.e^{y}.[e^{z}]_{0}^{x+y}dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x} e^{x}.e^{y}.(e^{x+y}-e^{0})dy\,dx$

$$\Rightarrow I=\int_{x=0}^{a}\int_{y=0}^{x}(e^{2x}.e^{2y}-e^{x}.e^{y})dy\,dx=\int_{x=0}^{a}\left[e^{2x}\frac{e^{2y}}{2}-e^{x}.e^{y}\right]_{y=0}^{x}dx$$

$$\Rightarrow I=\int_{x=0}^{a}\left[\left(\frac{e^{4x}}{2}-e^{2x}\right)-\left(\frac{e^{2x}}{2}-e^{x}\right)\right]dx=\int_{x=0}^{a}\left(\frac{e^{4x}}{2}-\frac{3}{2}e^{2x}+e^{x}\right)dx$$

$$\Rightarrow I=\left(\frac{e^{4x}}{8}-\frac{3e^{2x}}{4}+e^{x}\right)_{0}^{a}=\left(\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^{a}\right)-\left(\frac{1}{8}-\frac{3}{4}+1\right)$$

$$\Rightarrow I=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^{a}-\frac{3}{8}\quad \textbf{OR}\ \mathbf{I=\frac{1}{8}(e^{4a}-6e^{2a}+8e^{a}-3)}$$

4. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$ **[V.T.U-2013, Jan-2018]**

Sol. Let $I=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

$$I=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}(xy)\left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{1-x^2-y^2}}dy\,dx=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}(xy)\frac{(1-x^2-y^2)}{2}dy\,dx$$

$$\Rightarrow I=\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}(xy-x^3y-xy^3)dy\,dx=\frac{1}{2}\int_{x=0}^{1}\left[\frac{xy^2}{2}-\frac{x^3y^2}{2}-\frac{xy^4}{4}\right]_{y=0}^{\sqrt{1-x^2}}dx$$

$$\Rightarrow I=\frac{1}{2}\int_{x=0}^{1}\left[\frac{x(1-x^2)}{2}-\frac{x^3(1-x^2)}{2}-\frac{x(1-x^2)^2}{4}\right]dx$$

$$\Rightarrow I=\frac{1}{2}\int_{x=0}^{1}\left[\frac{2x-2x^3-2x^3+2x^5-x-x^5+2x^3}{4}\right]dx=\frac{1}{8}\int_{x=0}^{1}(x-2x^3+x^5)dx$$

$$\Rightarrow I=\frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_{x=0}^{1}=\frac{1}{8}\left[\frac{1}{2}-\frac{2}{4}+\frac{1}{6}\right]=\frac{1}{8}\left[\frac{3-3+1}{6}\right]=\mathbf{\frac{1}{48}}$$

5. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(1+x+y+z)^3}dx\,dy\,dz$. **[June-2008, Jan-2005]**

Sol. Let $I=\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}\frac{1}{(1+x+y+z)^3}dz\,dy\,dx=\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{-1}{2(1+x+y+z)^2}\right]_{z=0}^{1-x-y}dy\,dx$

$$\Rightarrow I=\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{-1}{8}+\frac{1}{2(1+x+y)^2}\right]dy\,dx=\int_{x=0}^{1}\left[-\frac{1}{8}y-\frac{1}{2(1+x+y)}\right]_{y=0}^{1-x}dx$$

$$\Rightarrow I=\int_{x=0}^{1}\left[-\frac{1}{8}(1-x)-\frac{1}{4}+\frac{1}{2(1+x)}\right]dx=\int_{x=0}^{1}\left[-\frac{3}{8}+\frac{x}{8}+\frac{1}{2(1+x)}\right]dx$$

$$\Rightarrow I=\left[-\frac{3x}{8}+\frac{x^2}{16}+\frac{1}{2}\log(1+x)\right]_{x=0}^{1}=\left[-\frac{3}{8}+\frac{1}{16}+\frac{1}{2}\log 2\right]=\frac{-5}{16}+\frac{1}{2}\log 2$$

OR $\mathbf{I=\log\sqrt{2}-\frac{5}{16}}$.

6. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\left(\frac{a^2-r^2}{a}\right)} r\,dr\,d\theta\,dz$ **[July-2005, 07, 09]**

Sol. Let $I=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{\left(\frac{a^2-r^2}{a}\right)} r\,dz\,dr\,d\theta$

$$\Rightarrow I=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r[z]_0^{\left(\frac{a^2-r^2}{a}\right)}dr\,d\theta=\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a\sin\theta}\left[r\left(\frac{a^2-r^2}{a}\right)\right]dr\,d\theta$$

$$\Rightarrow I = \frac{1}{a}\int_{\theta=0}^{\frac{\pi}{2}}\left[a^2\left(\frac{r^2}{2}\right) - \frac{r^4}{4}\right]_{r=0}^{a\sin\theta} d\theta = \frac{1}{a}\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{a^4}{2}\sin^2\theta - \frac{a^4}{4}\sin^4\theta\right] d\theta$$

Using $\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\dots\dots\frac{1}{2}\frac{\pi}{2}$ (when n is even)

$$\therefore I = \frac{1}{a}\left\{\frac{a^4}{2}\frac{1}{2}\frac{\pi}{2} - \frac{a^4}{4}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right\} = \frac{\pi a^4}{8a}\left(1-\frac{3}{8}\right) = \boldsymbol{\frac{5\pi a^3}{64}}$$

7. Evaluate $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$ **[Jan-2017]**

Sol. Let $I = \int_{z=0}^4\int_{x=0}^{2\sqrt{z}}\int_{y=0}^{\sqrt{4z-x^2}} dy\, dx\, dz$

$$I = \int_{z=0}^4\int_{x=0}^{2\sqrt{z}}[y]_0^{\sqrt{4z-x^2}} dx\, dz = \int_{z=0}^4\int_{x=0}^{2\sqrt{z}}\sqrt{4z-x^2}\; dx\, dz$$

Let $4z = a^2$ (for convenient)so that $2\sqrt{z} = a$

$$I = \int_{z=0}^4\int_{x=0}^{a}\sqrt{a^2-x^2}\, dx\, dz = \int_{z=0}^4\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a} dz$$

$$I = \int_{z=0}^4 0 + \frac{a^2}{2}(\sin^{-1}1 - \sin^{-1}0)\, dz = \int_{z=0}^4\frac{a^2}{2}\left(\frac{\pi}{2}-0\right)dz = \int_{z=0}^4\frac{a^2\pi}{4}dz$$

$$I = \frac{\pi}{4}\int_{z=0}^{4} 4z\, dz = \pi\left[\frac{z^2}{2}\right]_0^4 = \pi\left(\frac{16}{2}\right) = \boldsymbol{8\pi}$$

Evaluation of $\iint f(x,y)dx\, dy$ over the specific region

We need to draw the befitting figure from the given description to identify the specific region R, We have to then express

$I = \iint f(x,y)dx\, dy = \int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)dy\, dx \dots\dots(1)$ OR

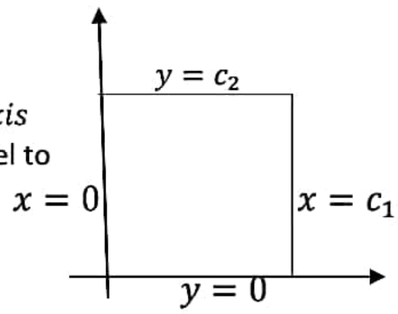
$I = \iint f(x,y)dx\, dy = \int_{y=a}^{b}\int_{x=x_1(y)}^{x_2(y)} f(x,y)dx\, dy \dots\dots(2)$

Note: Some of important and standard curves along with their equation and shape is given below as it will be highly useful for working problems.

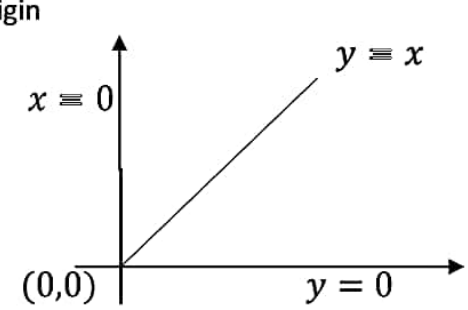
1. Straight Lines:

(i) $x = 0$ And $y = 0$ are respectively the equation of y & $x - axis$

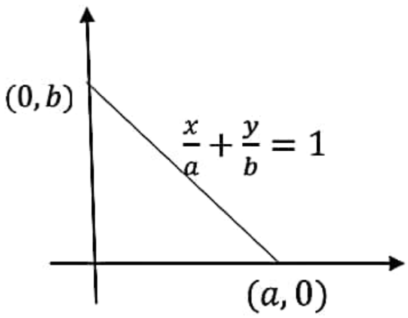
(ii) $x = c_1$&$y = c_2$ Are respectively the equation of a line parallel to Y-axis & a line parallel to x-axis



(iii) $y = x$ is a straight line passing through the origin

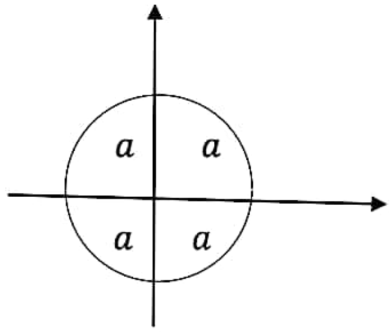


(iv) $\frac{x}{a}+\frac{y}{b}=1$ is a straight line having x intercept 'a' & y Intercept 'b' i.e. a straight line passing through $(a,0)$& $(y,0)$



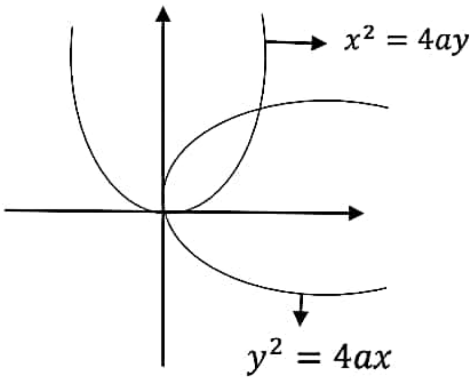
2. Circle:

$x^2+y^2=a^2$ Be a circle with center origin& radius 'a'



3. Parabola: $y^2=4ax$ is symmetrical about x-axis

$x^2=4ay$ is symmetrical about y-axis



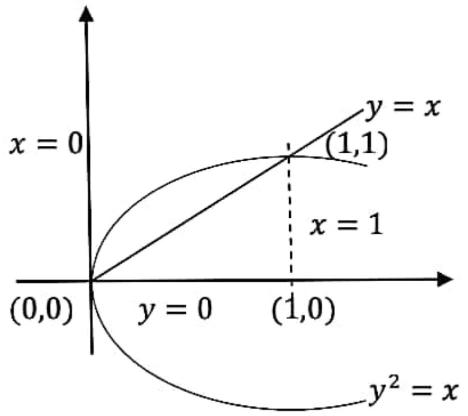
Evaluation of double integral by changing the order of integration:

Problems:

1. **Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dydx$ changing the order of integration** **[V.T.U-2010]**

Sol .

Given $x=0, x=1$ and $y=x,\ y=\sqrt{x}$ OR $y^2=x$



If $x=0 \Rightarrow y=0$ and $x=1 \Rightarrow y=1$ ∴(0,0)&(1,1)is the intersection point

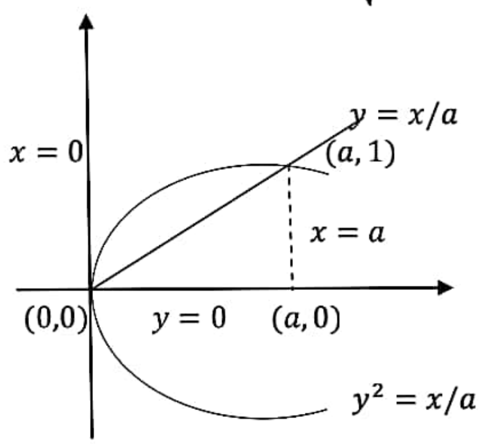
Let $I=\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} xy\, dydx,$ on changing order of integration

$$I=\int_{y=0}^{1}\int_{x=y^2}^{y} xy\, dxdy=\int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_{y^2}^{y} dy=\frac{1}{2}\int_{y=0}^{1} y(y^2-y^4)dy=\frac{1}{2}\int_{y=0}^{1}(y^3-y^5)dy$$

$$I=\frac{1}{2}\left[\frac{y^4}{4}-\frac{y^6}{6}\right]_0^1=\frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right)=\frac{1}{2}\left(\frac{1}{12}\right)=\mathbf{\frac{1}{24}}$$

2. **Evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dydx$ changing the order of integration** **[V.T.U-2013, 14]**

Sol . Given $x=0, x=a$ and $y=\frac{x}{a},\ y=\sqrt{\frac{x}{a}}$ OR $y^2=\frac{x}{a}$



If $x = 0 \Rightarrow y = 0$ and $x = a \Rightarrow y = 1$ $\therefore (0,0)\&(a,1)$is the intersection point

Let $I = \int_{x=0}^{a} \int_{y=x/a}^{\sqrt{x}/a} (x^2 + y^2)\, dydx$, on changing order of integration

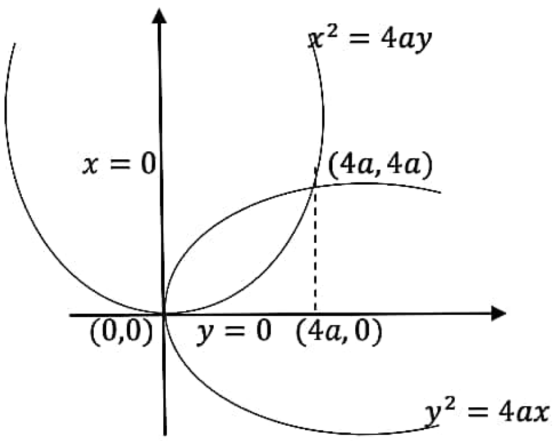
$I = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2 + y^2)\, dxdy$

$I = \int_{y=0}^{1} \left[\frac{x^3}{3} + y^2 x\right]_{ay^2}^{ay} dy = \int_{y=0}^{1} \left[\frac{1}{3}(a^3y^3 - a^3y^6) + (ay^3 - ay^4)\right] dy$

$I = \frac{a^3}{3}\left[\frac{y^4}{4} - \frac{y^7}{7}\right]_0^1 + a\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1 = \frac{a^3}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + a\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{a^3}{28} + \frac{a}{20}$

3. Evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} (xy)\, dydx$ changing the order of integration [V.T.U-2014, Jan-2017]

Sol. Given $x = 0\,, x = 4a$ and $y = \frac{x^2}{4a},\ y = 2\sqrt{ax}$ OR $x^2 = 4ay,\ \ y^2 = 4ax$



If $x = 0 \Rightarrow y = 0$ and $x = 4a \Rightarrow y = 4a$

$\therefore (0,0)\&(4a,4a)$is the intersection point

Let $I = \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} (x^2 + y^2)\, dydx$, on changing order of integration

$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} (xy)\, dxdy = \int_{y=0}^{4a} y \left[\frac{x^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy = \frac{1}{2}\int_{y=0}^{4a} y\left(4ay - \frac{y^4}{16a^2}\right) dy$

$I = \frac{1}{2}\int_{y=0}^{4a} \left(4ay^2 - \frac{y^5}{16a^2}\right) dy = \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^6}{96a^2}\right]_0^{4a} = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{4096a^6}{96a^2}\right]$

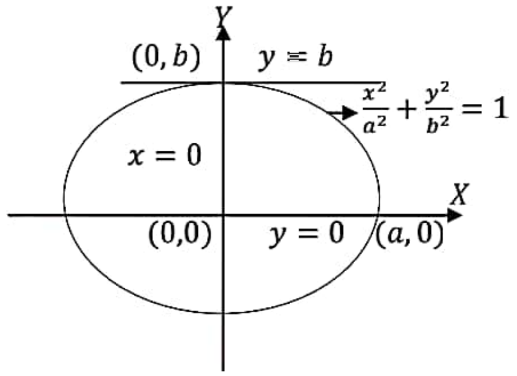
$I = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{128a^4}{3}\right] = \frac{64a^4}{3}$

4. Evaluate $\int_0^{b} \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} (xy)\, dxdy$ changing the order of integration [V.T.U-2011]

Sol. Given $y = 0,\ y = b$ and $x = 0, x = \frac{a}{b}\sqrt{b^2 - y^2}$ **OR** $x^2 = \frac{a^2}{b^2}(b^2 - y^2) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

If $y = 0 \Rightarrow x = a$ and $y = b \Rightarrow x = 0$

$\therefore$ The point of intersection $(a,0)\&(0,b)$



Let $I=\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}}(xy)dxdy$, on changing order of integration

$$I=\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}(xy)dydx=\int_{x=0}^{a}x\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx=\frac{1}{2}\int_{x=0}^{a}x\left[\frac{b^2}{a^2}(a^2-x^2)\right]dx$$

$$I=\frac{b^2}{2a^2}\int_{x=0}^{a}x(a^2-x^2)dx=\frac{b^2}{2a^2}\int_{x=0}^{a}(a^2x-x^3)dx=\frac{b^2}{2a^2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^{a}$$

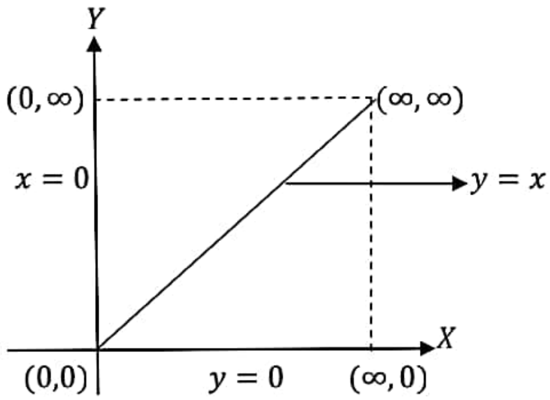
$$I=\frac{b^2}{2a^2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right)=\frac{a^4b^2}{8a^2}=\boldsymbol{\frac{a^2b^2}{8}}$$

5. Evaluate $\int_0^{\infty}\int_0^{x}xe^{-\frac{x^2}{y}}dydx$ changing the order of integration **[Jan-2017]**

Sol. Given $x=0, x=\infty$ and $y=0,\ y=x$

If $x=0\Rightarrow y=0$ and $x=\infty\Rightarrow y=\infty$

$\therefore (0,0)(\infty,0)\&(\infty,\infty)$ is the point of intersection



Let $I=\int_{x=0}^{\infty}\int_{y=0}^{x}xe^{-\frac{x^2}{y}}dydx$, on changing order of integration

$$I=\int_{y=0}^{\infty}\int_{x=y}^{\infty}xe^{-\frac{x^2}{y}}dxdy$$

Put $\frac{x^2}{y} = t \quad \therefore \frac{2x}{y} dx = dt$ or $x dx = y\frac{dt}{2}$

Also when $x = y, \ t = y$ and when $x = \infty, \ t = \infty$

$\therefore I = \int_{y=0}^{\infty}\int_{t=y}^{\infty} e^{-t}\frac{y}{2}\, dt\, dy = \int_{y=0}^{\infty}\frac{y}{2}[-e^{-t}]_y^{\infty}\, dy = -\frac{1}{2}\int_{y=0}^{\infty} y\,[e^{-\infty} - e^{-y}]\, dy$

$I = -\frac{1}{2}\int_{y=0}^{\infty} y\,[0 - e^{-y}]\, dy = \frac{1}{2}\int_{y=0}^{\infty} y\, e^{-y}\, dy$

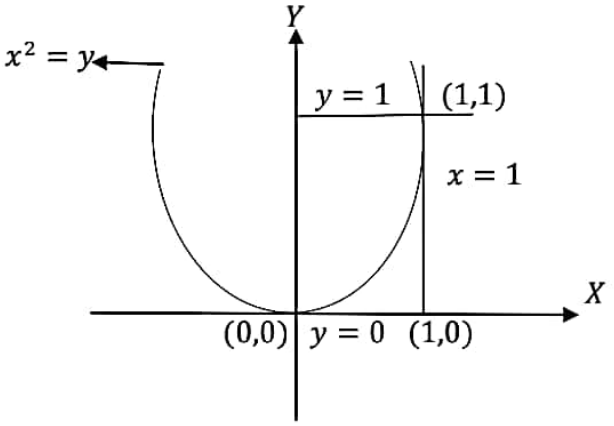
Applying Bernoulli's rule,

$I = \frac{1}{2}\{[y(-e^{-y})]_0^{\infty} - [(1)(e^{-y})]_0^{\infty}\} = \frac{1}{2}[0 - (0-1)] = \frac{1}{2} \quad (\because e^{-\infty} = 0)$

6. Evaluate $\int_0^1\int_{\sqrt{y}}^1 dxdy$ changing the order of integration

Sol. Given $y = 0, \ y = 1$ and $x = \sqrt{y}, \ x = 1$ OR $x^2 = y$

if $y = 0 \Rightarrow x = 0, \quad y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow x = 1$



$\therefore (0,0), (1,0) \& (1,1)$ is the point of intersection

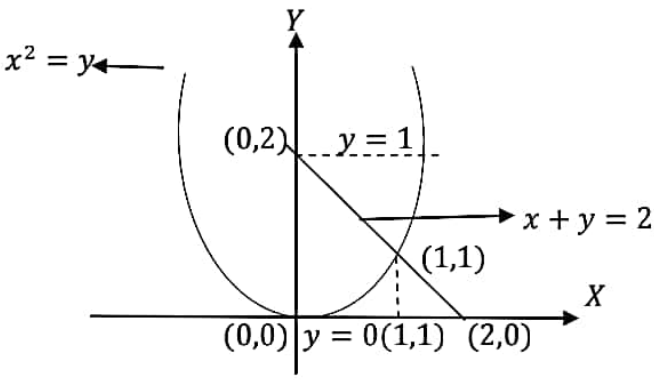
Let $I = \int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} dxdy$, on changing order of integration

$I = \int_{x=0}^{1}\int_{y=0}^{x^2} dydx = \int_{x=0}^{1}[y]_0^{x^2}\, dx = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

7. Evaluate $\int_0^1\int_{\sqrt{y}}^{2-y} xy\, dxdy$ changing the order of integration **[VTU-2006]**

Sol. Given $y = 0, \ y = 1$ and $x = \sqrt{y}, \ x = 2 - y$ OR $x^2 = y$ & $x + y = 2$

if $y = 0 \Rightarrow x = 0, \quad y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow x = 1$



$\therefore (0,0), (2,0) \& (1,1)$ is the point of intersection

Let $I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} xy\, dxdy$, on changing order of integration

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} xydydx + \int_{x=1}^{2} \int_{y=0}^{2-x} xydydx$$

$$I = \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_0^{x^2} dx + \int_{x=1}^{2} x\left[\frac{y^2}{2}\right]_0^{x^2} dx = \int_{x=0}^{1} \frac{x}{2}(x^4 - 0)dx + \int_{x=1}^{2} \frac{x}{2}[(2-x)^2 - 0]\, dx$$

$$I = \frac{1}{2}\int_{x=0}^{1} x^5 dx + \frac{1}{2}\int_{x=1}^{2} x(4 + x^2 - 4x)dx = \frac{1}{2}\int_{x=0}^{1} x^5 dx + \frac{1}{2}\int_{x=1}^{2} (x^3 - 4x^2 + 4x)dx$$

$$I = \frac{1}{2}\left[\frac{x^2}{6}\right]_0^1 + \frac{1}{2}\left[\frac{x^4}{4} - 4\frac{x^3}{3} + 4\frac{x^2}{2}\right]_1^2 = \frac{1}{12} + \frac{1}{2}\left[\frac{15}{4} - \frac{28}{3} + 6\right] = \frac{7}{24}$$

Evaluation by changing into polar co-ordinate

Note: By changing into polar co-ordinates we have to use some suitable substitution.

$i,e\;\; x = r\,cos\theta,\; y = r\,sin\theta$ and $x^2 + y^2 = r^2$ and $dxdy = r\, drd\theta$.

Problems:

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy$ by changing in to polar coordinates. [Jan-2017]

Hence show that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi/2}$.

Sol. Given $x = 0, x = \infty$ and $y = 0,\; y = \infty$

In polar coordinates we have $x = r\,cos\theta,\; y = r\,sin\theta$

$x^2 + y^2 = r^2$ and $dxdy = r\, drd\theta$

Since $x, y \to 0$ to ∞, then $r \to 0$ to ∞ also $\theta \to 0$ to $\frac{\pi}{2}$ in the first quadrate

Thus $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, rdrd\theta$, put $r^2 = t \Rightarrow 2rdr = dt$, or $rdr = \frac{dt}{2}$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}\, dt\, d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} [-e^{-t}]_0^\infty d\theta = \int_{\theta=0}^{\pi/2} -[e^{-\infty} - e^0]d\theta,$$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2} -[0 - 1]d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} d\theta = \frac{1}{2}[\theta]_0^{\pi/2} = \frac{\pi}{4}.$$

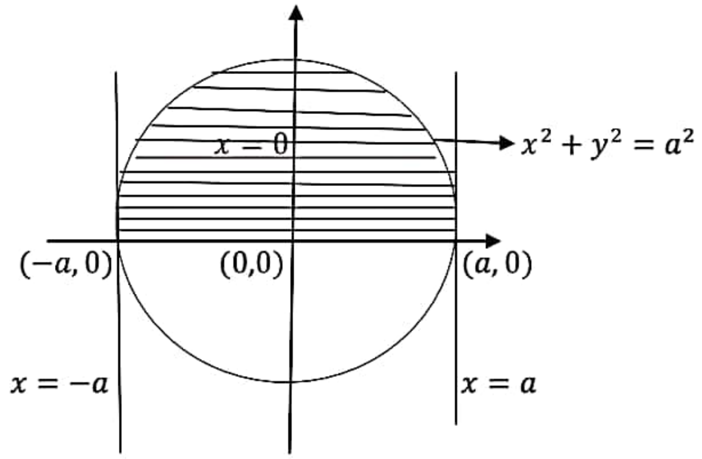
$$\int_0^\infty e^{-x^2} dx = \int_{t=0}^{\infty} e^{-t}\frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_{t=0}^{\infty} e^{-t} t^{-1/2} dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{\frac{1}{2}-1} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi/2}$$

$\because$ put $x^2 = t \Rightarrow 2xdx = dt$ OR $dx = \frac{dt}{2\sqrt{t}}$ and

$\because\; \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx,$

2. Evaluate $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dydx$ by changing in to polar coordinates.

Sol.



Here $x=-a, x=a$ and $y=0,\ y=\sqrt{a^2-x^2}$ OR $x^2+y^2=a^2$

The region of the integration is as in the figure

$\therefore$ $(0,0)(a,0)\&(-a,0)$ is the intersection point.

Clearly θ varies from 0 to π.

In polar coordinates we have $x=r\cos\theta,\ y=r\sin\theta$

$x^2+y^2=r^2$ and $dxdy=r\,drd\theta$

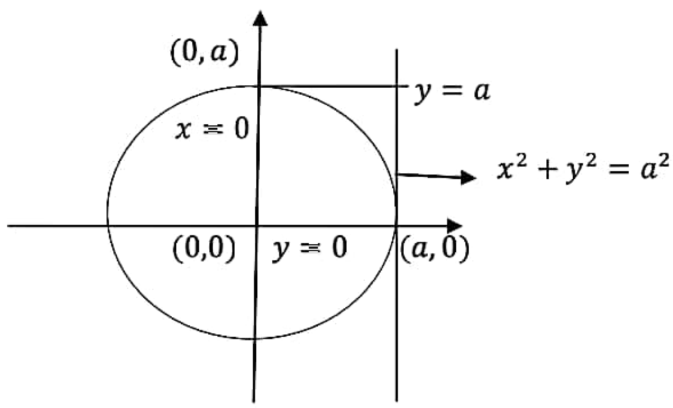
$i.e\ a^2=r^2 \Rightarrow r=a\ \ \therefore r\to 0$ to a Also $\theta \to 0$ to π.

$I=\int_{\theta=0}^{\pi}\int_{r=0}^{a} r.\,rdrd\theta=\int_{\theta=0}^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta=\frac{a^3}{3}[\theta]_0^{\pi}=\frac{\pi a^3}{3}$.

3. Evaluate $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} (x^2+y^2)\, dxdy$ by changing in to polar coordinates.

Sol. Here $y=0,\ y=a$ and $x=0,\ x=\sqrt{a^2-y^2}$ OR $x^2+y^2=a^2$

The region of the integration is as in the figure



$\therefore (0,0)(a,0)\&(0,a)$ is the intersection point.

Clearly θ varies from 0 to $\frac{\pi}{2}$.

In polar coordinates we have $x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2$ and $dxdy = r\,drd\theta$

$i.e\ a^2 = r^2 \Rightarrow r = a\ \ \therefore r \to 0$ to a Also $\theta \to 0$ to $\frac{\pi}{2}$.

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2.rdrd\theta = \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\right]_0^a d\theta = \frac{a^4}{4}[\theta]_0^{\pi/2} = \frac{\pi a^4}{8}.$$

Application of Double and Triple Integral

Note : 1. $\iint_R dx\,dy$ = Area of the region R in the Cartesian form .

2. $\iint_R r\,dr\,d\theta$ = Area of the region R in the Polar form.

3. $\iiint_V dx\,dy\,dz$ = Volume of the solid in the Cartesian form.

4. $\iint_A 2\pi r^2 \sin\theta\ dr\,d\theta$ = Volume of a solid obtained by the revolution of a curve Enclosing an area A about initial line in the polar form.

Problems

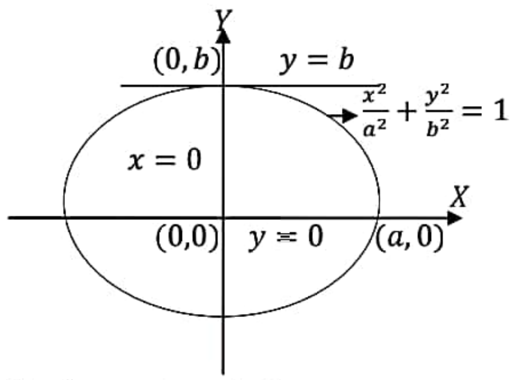
1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration. [Jan-2018]

Sol. Area A= $\iint_R dx\,dy$

Here x varies from 0 to a and y varies from 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$.

Now Area of the ellipse $=4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$

$$A = 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\ dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\ dx$$



$$I = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = \frac{4b}{a}\left[\left(0 + \frac{a^2}{2}.\frac{\pi}{2}\right) - (0 + 0)\right] = \boldsymbol{\pi ab} \text{ sq.units}$$

2. Find the area enclosed between the parabola $y = x^2$ and the straight line $y = x$

Sol. Given $y = x^2$ --------(1) and $y = x$----------(2)

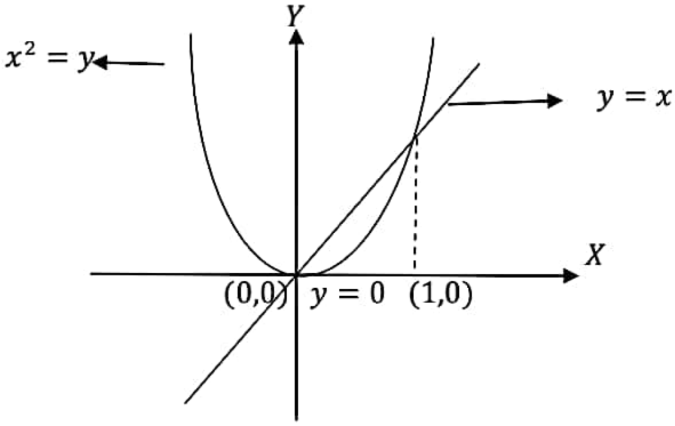
Subtracting eq.(2) from eq.(1) we get

$x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$

Now when $x = 0,\ y = 0$ and when $x = 1,\ y = 1$

$\therefore$ (0,0)&(1,1)is the intersection point.

So $x: 0 \to 1$ and $y: x^2 \to x$



$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 (x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$A = \frac{1}{2} - \frac{1}{3} = \frac{\mathbf{1}}{\mathbf{6}}$$

3. Find the area enclosed by the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

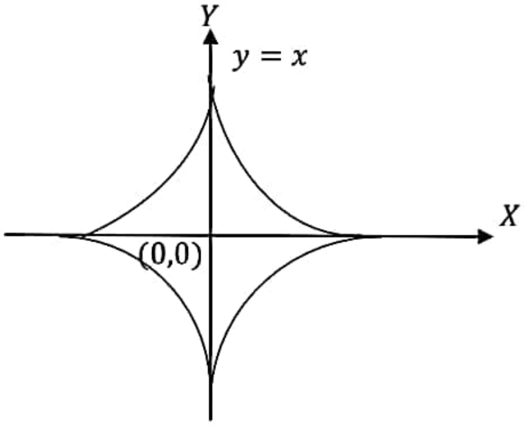
Sol. We have that x varies from 0 to a and y varies from 0 to $\left(a^{2/3} - x^{2/3}\right)^{3/2}$.

Therefore required area $4\int_0^a \int_0^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} dy\,dx = 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$

Put $x = a\,sin^3\theta$. then θ varies from 0 to $\frac{\pi}{2}$ and $dx = 3a\,sin^2\theta\,cos\,\theta d\theta$. so

Area $= 4\int_0^{\pi/2} \left(a^{2/3} - a^{2/3}\sin^2\theta\right)^{3/2} 3a\,sin^2\theta\,cos\,\theta d\theta. = 3 \times 4a\int_0^{\pi/2} \cos^4\theta \sin^2\theta d\theta$

$= 12a^2 \frac{1\times3\times1}{6\times4\times2}\frac{\pi}{2} = \frac{3}{8}\pi a^2$ sq.units



4. Find the area inside the circle $r = a\ sin\theta$ but lying outside the cardioids $r = a(1 - cos\ \theta)$. **[Jan-2017]**

Sol. The region of area is bounded by $r = a\ sin\theta \ldots\ldots\ldots..(1)$

$r = a\ (1 - cos\theta) \ldots\ldots\ldots..(2)$

We find the point of intersection of curve: From Equations (1) and (2)

$$a\ sin\theta = a(1 - cos\theta) \Rightarrow a\sin\theta = 2a\sin^2\frac{\theta}{2} \Rightarrow \frac{2a\sin^2(\frac{\theta}{2})}{2a\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})} = 1$$

$$\Rightarrow \tan\frac{\theta}{2} = 1 \Rightarrow \frac{\theta}{2} = \tan^{-1}(1) \Rightarrow \frac{\theta}{2} = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}.$$

Therefore the point of intersection are $(0,0)\&\left(a,\frac{\pi}{2}\right)$.

See in figure for $r = a\sin\theta$, which is a circle , we have

$$r = \frac{a}{r}\ r\sin\theta \Rightarrow r^2 = ay \Rightarrow x^2 + y^2 - ay = 0 \Rightarrow x^2\left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

That is ,it is a circle with centre are $\left(0,\frac{a}{2}\right)$and radius $\frac{a}{2}$.

Area $=\int_0^{\pi/2}\int_{a\sin\theta}^{a(1-\cos\theta)} r\ dr\ d\theta =$

Area$=\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sin\theta}^{a(1-\cos\theta)} d\theta = \frac{1}{2}\int_0^{\pi/2}[a^2(1-\cos\theta\ )^2 - a^2\sin^2\theta]d\theta$

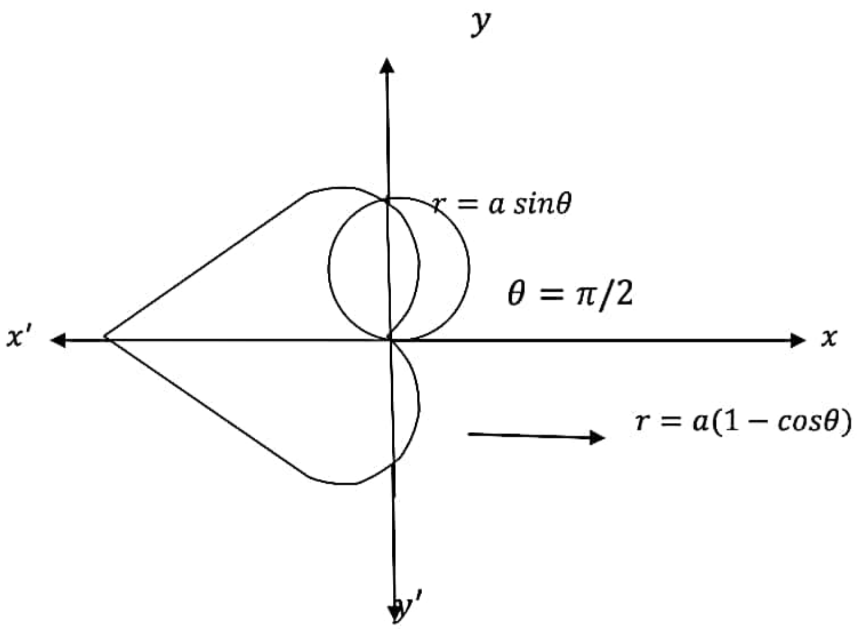
Area $= \frac{a^2}{2}\int_0^{\pi/2}[(1-\cos\theta\ )^2 - (1-\cos^2\theta)]d\theta$

Area $= \frac{a^2}{2}\int_0^{\pi/2}[(1-\cos\theta\ )^2 - (1-cos\ \theta)(1+cos\theta)]d\theta$

Area $= \frac{a^2}{2}\int_0^{\pi/2}(1-\cos\theta)[1-\cos\theta-1-\cos\theta]d\theta = \frac{a^2}{2}\int_0^{\pi/2}(1-\cos\theta)(-2\cos\theta)d\theta$

Area $= a^2\int_0^{\pi/2}(\cos^2\theta - \cos\theta)d\theta = a^2\left\{\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - \int_0^{\pi/2}\cos\theta d\theta\right\}$

Area $= a^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} - a^2[\sin\theta]_0^{\pi/2} = \frac{a^2}{4}(\pi - 4)$ sq. units



Beta and Gamma Function:

Beta and Gamma function which helps to evaluate certain definite integrals which are either Difficult or impossible to evaluate by various known methods.

Definition: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad (m,n > 0)$ is called the Beta function.

$\Gamma(\text{n}) = \int_0^\infty e^{-x}x^{n-1}\,dx, \quad (n > 0)$Is called the Gamma function

Alternative definition Beta and Gamma function

$\beta(m,n) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$ And

$\Gamma(n) = 2\int_{\theta=0}^{\infty} e^{-x^2}x^{2n-1}\,dx.$

Note: 1.$\Gamma(n+1) = n\Gamma(n)$

2.$\Gamma(n) = (n-1)\Gamma(n-1)$

3. $\Gamma(n-1) = (n-2)\Gamma(n-2)$

Properties of Beta and Gamma Function

1. Show that $\boldsymbol{\beta(m,n) = \beta(n,m)}$

Sol. We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad (m,n > 0)$

put $x = 1-y \Rightarrow dx = -dy$

If $x = 0 \Rightarrow y = 1$ and if $x = 1 \Rightarrow y = 0$

$\beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1} - dy = -\int_1^0 (1-y)^{m-1}(y)^{n-1}dy$

$\beta(m,n) = \int_0^1 (y)^{n-1}(1-y)^{m-1}dy = \beta(n,m)$

2. S.T (*i*)$\Gamma(n+1) = n\Gamma(n)$, (*ii*)$\Gamma(n+1) = n!$

Proof. We know that by definition $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$, $(n > 0)$

$(i)\Gamma(n+1) = \int_0^\infty e^{-x}x^n\,dx$

Apply integrating by parts

$\Gamma(n+1) = x^n\left[\frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty \frac{e^{-x}}{-1}\,nx^{n-1}\,dx$

$\Gamma(n+1) = (0-0) + n\int_0^\infty e^{-x}x^{n-1}\,dx = n\Gamma(n)$ $(\because e^{-\infty} = 0)$

$\therefore \Gamma(n+1) = n\Gamma(n)$

(ii) consider $\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+1) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$ so on

$\Gamma(n+1) = n(n-1)(n-2)\ldots\ldots.3.2.1\;\Gamma(1) = n!$

But $\Gamma(1) = \int_0^\infty e^{-x}x^0\,dx = [-e^{-x}]_0^\infty = 1$

Thus $\Gamma(n+1) = n!$

Relation between Beta Gamma Functions:

3. To prove that $\boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$ **[Dec-2010, June-2013, 16, 17, 18]**

Proof. Let $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx \ldots.(1)$

Put $n = m$ and $x = y$ in (1) we get

$\Gamma(\mathrm{m}) = 2\int_0^\infty e^{-y^2}y^{2m-1}\,dy \ldots.(2)$

$\beta(m,n) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \ldots..(3)$

Consider $\Gamma(\mathrm{m})\Gamma(\mathrm{n}) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,x^{2n-1}y^{2m-1}\,dx\,dy\,\left(\because e^{-x^2}e^{-y^2} = e^{-(x^2+y^2)}\right)$

Let us evaluate by changing into polar co-ordinates

Put $x = r\cos\theta$ $y = r\sin\theta$

And $x^2 + y^2 = r^2$, $dx\,dy = r\,dr\,d\theta$

$\therefore r \to 0\ to\ \infty$, $\theta \to 0\ to\ \frac{\pi}{2}$

$\Rightarrow \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = 4\int_{r=0}^\infty\int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}\,r\,dr\,d\theta$

$\Rightarrow \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = \left\{2\int_{r=0}^\infty e^{-r^2}r^{2m+2n-1}\,dr\right\}\left\{2\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2n-1}\theta\sin^{2m-1}\theta\,d\theta\right\}$

$\Rightarrow \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = \left\{2\int_{r=0}^\infty e^{-r^2}r^{2(m+n)-1}\,dr\right\}\beta(m,n)$ $(\because \text{using}(3))$

$\therefore \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = \Gamma(m+n)\beta(m,n)(\because \Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx)$

Thus $\boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$

4. Prove that $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$ **[Jan-2016]**

Sol. We have $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \ldots..(1)$

Put $m = n = \frac{1}{2}$ in (1) we get

$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \dots (2)$

We know that $\beta(m,n) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$, put $m = n = \frac{1}{2}$

$\Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta\cos^0\theta\, d\theta = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2}\right] = \pi \dots (3)$

Substitute (3) in (2) we get

$\pi = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Duplication Formula in terms of Beta and Gamma Function:

5. Prove that $(i)\beta\left(m,\frac{1}{2}\right) = 2^{2m-1}\beta(m,m)\,(ii)\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

Sol. $\beta(m,n) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \dots (1),\ n = \frac{1}{2}$

$\beta\left(m,\frac{1}{2}\right) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^0\theta\, d\theta = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\, d\theta \dots\dots (2)$

put $n = m$ in (1)

$\beta(m,m) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2m-1}\theta\, d\theta = 2\int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$

$(\because \frac{\sin 2\theta}{2} = \sin\theta\, cos\theta)$

Put $2\theta = \phi \Rightarrow 2d\theta = d\phi$, $\quad if\ \theta = 0 \Rightarrow \phi = 0$

$\theta = \frac{\pi}{2} \Rightarrow \phi = \pi$

$\beta(m,m) = \frac{1}{2^{2m-1}}\int_{\phi=0}^{\pi} \sin^{2m-1}\phi\, d\phi = \frac{1}{2^{2m-1}} 2\int_{\phi=0}^{\pi/2} \sin^{2m-1}\phi\, d\phi$

$\therefore \beta\left(m,\frac{1}{2}\right) = 2^{2m-1}\beta(m,m)(\because using\ (2)$

(ii) We have $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \dots (1)$

Put LHS $n = \frac{1}{2}$ and RHS $n = m$ in (1) we get

$\beta\left(m,\frac{1}{2}\right) = \frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)}$

$\Rightarrow \frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}\frac{\Gamma(m)}{\Gamma(2m)}$

$\Rightarrow \Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

Note: $\beta(m,n) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \dots (1)$

Put $2m-1 = p \qquad 2n-1 = q$

$\Rightarrow m = \frac{p+1}{2} \qquad n = \frac{q+1}{2}$

$$\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = 2\int_{\theta=0}^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta$$

1. $\int_{\theta=0}^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

2. $\int_{\theta=0}^{\frac{\pi}{2}} \sin^{p-1}\theta \cos^{q-1}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p}{2},\frac{q}{2}\right)$

3. $\int_0^1 x^m(1-x)^n dx = \beta(m+1,n+1)$

4. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m) \dots (2)$

Put $m = \frac{1}{4}$ in (2) we get

$\Gamma(1/4)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2\frac{1}{4}-1}}\Gamma\left(2\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{2^{-\frac{1}{2}}}\sqrt{\pi}$

5. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

6. $\beta(m,n) = \beta(n,m)$, **by property**

7. Two important forms along with substitution is as fallow

(i) $(a-x^n)$: **Substitution :** $x^n = a\,sin^2\,\theta$

(ii) $(a+x^n)$: **Substitution :** $x^n = a\,tan^2\,\theta$

Problems:

1. Show That $\Gamma(n) = \int_0^1 \left(log\frac{1}{y}\right)^{n-1} dy \quad (n>0)$

Sol. Consider RHS= $\int_0^1 \left(log\frac{1}{y}\right)^{n-1} dy$, put $\log\frac{1}{y} = t \Rightarrow \frac{1}{y} = e^t$ or $y = e^{-t} \therefore dy = -e^{-t}dt$

If $y = 0 \Rightarrow e^{-t} = 0 \Rightarrow t = \infty$

If $y = 1 \Rightarrow t = 0$

$\therefore \text{RHS} = \int_0^1 \left(log\frac{1}{y}\right)^{n-1} dy = \int_\infty^0 t^{n-1}(-e^{-t})dt = -\int_\infty^0 e^{-t}t^{n-1}dt$

$\int_0^1 \left(log\frac{1}{y}\right)^{n-1} dy = \int_0^\infty e^{-t}t^{n-1}dt = \Gamma(n) = LHS$

2. Show That $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Sol. Consider RHS= $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$, put $x = \tan^2\theta \Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$

If $x = 0 \Rightarrow \tan^2\theta = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$

If $x = \infty \Rightarrow \tan^2\theta = \infty \Rightarrow \theta = \tan^{-1}(\infty) = \frac{\pi}{2}$

$\therefore RHS = \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1}.2\tan\theta\sec^2\theta}{(\sec^2\theta)^{m+n}}d\theta = 2\int_0^{\pi/2}\frac{\tan^{2m-2+1}\theta}{\sec^{2m+2n-2}\theta}d\theta$

$= \int_0^{\pi/2}\frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta}\cos^{2m+2n-2}\theta d\theta = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \beta(m,n) = LHS$

3. Show That $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ **[V.T.U-2011, 07, 08, Jan-17]**

Sol. We Know That $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

i.e. $\beta(m,n) = I_1 + I_2$

Consider $I_2 = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$, put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2}dy$

If $x = 1 \Rightarrow \frac{1}{y} = 1 \Rightarrow y = 1$

$x = \infty \Rightarrow \frac{1}{y} = \infty \Rightarrow y = \frac{1}{\infty} = 0$

$$\therefore I_2 = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{m-1}\frac{1}{y^2}}{\left(1+\frac{1}{y}\right)^{m+n}}\,dy = \int_0^1 \frac{\frac{1}{y^{m+1}}}{\left(\frac{y+1}{y}\right)^{m+n}}dy$$

$$\Rightarrow \int_0^1 \frac{\frac{y^{m+n}}{y^{m+1}}}{(1+y)^{m+n}}dy = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy,$$ replace $y \longrightarrow x$ we get

$$I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

4. Express $\int_0^1 x^m (1-x^n)^p\,dx$ in terms of Beta functions and hence Evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$. **[Jan-2017]**

Sol. Let $I = \int_0^1 x^m (1-x^n)^p\,dx$

Put $x^n = \sin^2\theta$ OR $x = \sin^{2/n}\theta \quad \therefore \quad dx = \frac{2}{n}\,.\sin^{\left(\frac{2}{n}\right)-1}\theta\cos\theta\,d\theta$

If $x = 0 \Rightarrow \sin^2\theta = 0$ *OR* $\theta = \sin^{-1}(0) \Rightarrow \theta = 0$

If $x = 1 \Rightarrow \sin^2\theta = 1$ *OR* $\theta = \sin^{-1}(1) \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \sin^{2m/n}\theta\cos^{2p}\theta.\frac{2}{n}.\sin^{\left(\frac{2}{n}\right)-1}\theta\cos\theta\,d\theta$$

$$I = \frac{2}{n}\int_0^{\pi/2} \sin^{\left(\frac{2m}{n}+\frac{2}{n}-1\right)}\theta\cos^{2p+1}\theta d\theta.$$

$$I = \frac{2}{n}.\frac{1}{2}\beta\left(\frac{\frac{2m}{n}+\frac{2}{n}-1+1}{2}, \frac{2p+1+1}{2}\right) \quad \because \int_{\theta=0}^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$I = \frac{2}{n}.\frac{1}{2}\beta\left(\frac{2\left(\frac{m}{n}+\frac{1}{n}\right)}{2}, \frac{2(p+1)}{2}\right)$$

Hence $\boldsymbol{I = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)}$

And $I = \int_0^1 x^5(1-x^3)^{10}\,dx.$

put $x^3 = t \Longrightarrow 3x^2dx = dt$ or $dx = \frac{dt}{3x^2}$

If $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1$

$$\therefore I = \int_0^1 x^5(1-t)^{10}\,\frac{dt}{3x^2} = \frac{1}{3}\int_0^1 t(1-t)^{10}\,dt = \frac{1}{3}\beta(1+1, 10+1)$$

$$\left(\because \int_0^1 x^m(1-x)^n dx = \beta(m+1, n+1)\right)$$

$$i.e\ I = \frac{1}{3}\beta(2,11) = \frac{1}{3}\left[\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)}\right] = \frac{1}{3}\left[\frac{1!10!}{12!}\right] = \frac{\mathbf{1}}{\mathbf{396}} \qquad (\because \Gamma(n+1) = n!)$$

5. S.T $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}\,\beta(p,q)$ [June-2011]

Sol. Let $I = \int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx$,

Put $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta = -4\sin\theta\cos\theta\, d\theta$

$If\ x = -1 \Rightarrow \cos 2\theta = -1 \Rightarrow 2\theta = \cos^{-1}(-1) = \pi \Rightarrow \theta = \frac{\pi}{2}$

$If\ x = 1 \Rightarrow \cos 2\theta = 1 \Rightarrow 2\theta = \cos^{-1}(1) = 0 \Rightarrow \theta = 0$

$\therefore I = \int_{\pi/2}^{0}(1+\cos 2\theta)^{p-1}\,(1-\cos 2\theta)^{q-1}\,(-4\sin\theta\cos\theta)\,d\theta$

$I = \int_{0}^{\pi/2}(2\cos^2\theta)^{p-1}\,(2\sin^2\theta)^{q-1}\,2^2\sin\theta\cos\theta\,d\theta$, on simplifying

$I = 2^{p+q}\int_{0}^{\pi/2}\cos^{2p-1}\sin^{2q-1}\theta d\theta = 2^{p+q}\frac{1}{2}\,\beta(q,p) = 2^{p+q-1}\beta(p,q)$

6. Express $\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$ in terms of gamma functions. [Jan-2014]

Sol. Let $I = \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$, put $x^4 = t$ $\therefore x = t^{1/4} \Rightarrow dx = \frac{1}{4}t^{-3/4}dt$

When $x = 0 \Rightarrow t = 0$; $x = 1 \Rightarrow t = 1$

$\therefore I = \int_0^1 \frac{1}{\sqrt{1-t}}\frac{1}{4}t^{-3/4}dt = \frac{1}{4}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-1/2}dt$

$I = \frac{1}{4}\,\beta\left(-\frac{3}{4}+1, -\frac{1}{2}+1\right)$ $\because \int_0^1 x^m(1-x)^n dx = \beta(m+1, n+1)$

$\therefore I = \frac{1}{4}\,\beta\left(\frac{1}{4},\frac{1}{2}\right)$

$\Rightarrow I = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ $\left(\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$

7. Evaluate $\int_0^4 x^{\frac{3}{2}}(4-x)^{5/2}dx$ by using Beta and Gamma functions. [Jan-2017]

Sol. Put $x = 4t \Rightarrow dx = 4dt$

If $x = 0 \Rightarrow t = 0$

If $x = 4 \Rightarrow t = 1$

$I = \int_0^1 (4t)^{\frac{3}{2}}(4-4t)^{5/2}\,4dt = \int_0^1 4^{\frac{3}{2}}.\,t^{\frac{3}{2}}.\,4^{\frac{5}{2}}.\,(1-t)^{\frac{5}{2}}.4\,dt$

$I = 4^5\int_0^1 t^{\frac{3}{2}}.(1-t)^{\frac{5}{2}}\,dt$

$I = 4^5\,\beta\left(\frac{3}{2}+1, \frac{5}{2}+1\right)$ $\because \int_0^1 x^m(1-x)^n dx = \beta(m+1, n+1)$

$I = 4^5\,\beta\left(\frac{5}{2},\frac{7}{2}\right) = \frac{4^5\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{7}{2}\right)} = \frac{4^5\,\frac{3}{2}.\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(6)} = \frac{1440\sqrt{\pi}\sqrt{\pi}}{5!} = \mathbf{12\pi}$

8. Evaluate $\int_0^1 \frac{x^3}{\sqrt{1-x}}\,dx$ by using Beta and Gamma functions.

Sol. $I = \int_0^1 x^3(1-x)^{-1/2}\,dx = \beta\left(3+1, -\frac{1}{2}+1\right) = \beta\left(4,\frac{1}{2}\right)$

$\because \int_0^1 x^m(1-x)^n dx = \beta(m+1, n+1)$

$I = \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} = \frac{3!\,\sqrt{\pi}}{\Gamma\left(\frac{9}{2}\right)} = \frac{3!\,\sqrt{\pi}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)} = \mathbf{\frac{32}{35}}$

9. Show That $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta = \pi$ [Jan-2015, 18, June-2012]

Sol. Let LHS= $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta d\theta$

Here $p = \frac{1}{2},\ q = 0; \frac{p+1}{2} = \frac{3}{4},\ \frac{q+1}{2} = \frac{1}{2}$ and $p = -\frac{1}{2},\ q = 0; \frac{p+1}{2} = \frac{1}{4},\ \frac{q+1}{2} = \frac{1}{2}$

$\because \int_{\theta=0}^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\therefore I = \frac{1}{2}\,\beta\left(\frac{3}{4},\frac{1}{2}\right) \times \frac{1}{2}\,\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right] = \frac{1}{4}\left[\sqrt{\pi}\Gamma\left(\frac{1}{4}\right) \times \frac{\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}\right] = \boldsymbol{\pi} = RHS$

$\left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{and } \Gamma(n+1) = n\Gamma(n)\right]$

10. Show That $\int_0^{\infty} xe^{-x^8}\, dx \times \int_0^{\infty} x^2 e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}$ [Jan-2013, 02, 01]

Sol. Let $I_1 = \int_0^{\infty} xe^{-x^8}\, dx$

Put $x^8 = t \therefore 8x^7\, dx = dt \text{ or } x\, dx = \frac{dt}{8x^6} = \frac{dt}{8\left(t^{\frac{1}{8}}\right)^6} = \frac{dt}{8\,t^{3/4}}; t \longrightarrow 0 \text{ to } \infty$

$\therefore I_1 = \int_{t=0}^{\infty} e^{-t} \frac{dt}{8\,t^{3/4}} = \frac{1}{8}\int_0^{\infty} e^{-t}t^{-3/4}dt$

We have $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}\, dx, \quad (n > 0)$, taking $n - 1 = -\frac{3}{4} \Rightarrow n = \frac{1}{4}$

Hence $I_1 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right) \ldots..(1)$

Let $I_2 = \int_0^{\infty} x^2 e^{-x^4}\, dx,$

Put $x^4 = t \therefore 4x^3 dx = dt \text{ or } x^2\, dx = \frac{dt}{4x} = \frac{dt}{4t^{1/4}}; t \longrightarrow 0 \text{ to } \infty$

$\therefore I_2 = \int_{t=0}^{\infty} e^{-t} \frac{dt}{4\,t^{1/4}} = \frac{1}{4}\int_0^{\infty} e^{-t}t^{-1/4}dt$

We have $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}\, dx, \quad (n > 0)$, taking $n - 1 = -\frac{1}{4} \Rightarrow n = \frac{3}{4}$

Hence $I_2 = \frac{1}{4}\Gamma\left(\frac{3}{4}\right) \ldots..(2)$

Equation (1) and (2) we get $I_1 \times I_2 = \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{32}\,\pi\sqrt{2} = \frac{\pi}{16\sqrt{\pi}}$

11. Show That $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^{\infty} x^2 e^{-x^4}\, dx = \frac{\pi}{4\sqrt{2}}$

Sol. . Let $I_1 = \int_0^{\infty} x^{1/2}e^{-x^2}\, dx$

Put $x^2 = t$ or $x = t^{1/2}$ or $\therefore 2xdx = dt \quad dx = \frac{dt}{2x} = \frac{dt}{2t^{1/2}}; t \longrightarrow 0 \text{ to} \infty$

$\therefore I_1 = \int_{t=0}^{\infty} e^{-t}t^{1/4} \frac{dt}{2\,t^{1/2}} = \frac{1}{2}\int_0^{\infty} e^{-t}t^{-3/4}dt$

We have $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}\, dx, \quad (n > 0)$, taking $n - 1 = -\frac{3}{4} \Rightarrow n = \frac{1}{4}$

Hence $I_1 = \frac{1}{2}\Gamma\left(\frac{1}{4}\right) \ldots..(1)$

Let $I_2 = \int_0^\infty x^2 e^{-x^4}\, dx,$

Put $x^4 = t \therefore 4x^3 dx = dt \quad or\ x^2\, dx = \frac{dt}{4x} = \frac{dt}{4t^{1/4}}; t \longrightarrow 0\ to\ \infty$

$\therefore I_2 = \int_{t=0}^\infty e^{-t} \frac{dt}{4\, t^{1/4}} = \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4} dt$

We have $\Gamma(\text{n}) = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad (n > 0)$, taking $n - 1 = -\frac{1}{4} \Rightarrow n = \frac{3}{4}$

Hence $I_2 = \frac{1}{4}\Gamma\left(\frac{3}{4}\right) \ldots..(2)$

Equation (1) and (2) we get $I_1 \times I_2 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{8}\, \pi\sqrt{2} = \frac{\pi}{4\sqrt{\pi}}$

12. Prove That $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}} dx = \frac{\pi}{4\sqrt{2}}$ [June-2014, Jan-2017]

Sol. Let $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx,$ put $x^4 = t \therefore x = t^{1/4} \Rightarrow dx = \frac{1}{4}t^{-3/4}dt$

When $x = 0 \Rightarrow t = 0; \quad x = 1 \Rightarrow t = 1$

$\therefore I_1 = \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} \frac{1}{4} t^{-3/4} dt = \frac{1}{4}\int_0^1 t^{-\frac{1}{4}}(1-t)^{-1/2} dt$

$\therefore I_1 = \frac{1}{4}\, \beta\left(-\frac{3}{4} + 1, -\frac{1}{2} + 1\right) \quad \left(\because \int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1)\right)$

$\therefore I_1 = \frac{1}{4}\, \beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$\Rightarrow I = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{3}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}\left(\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$And $\Gamma(n+1) = n\Gamma(n)$

$\therefore \boldsymbol{I_1} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \ldots..(1)$

Let $I_2 = \int_0^1 \frac{1}{\sqrt{1+x^4}} dx,$ put $x^2 = \tan\theta \Rightarrow dx = \frac{\sec^2\theta . d\theta}{2\sqrt{\tan\theta}}$

If $x = 0 \Rightarrow \theta = 0; \quad x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$I_2 = \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}\, 2\sqrt{\tan\theta}} d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta\, 2\sqrt{\tan\theta}} d\theta = \int_0^{\pi/4} \frac{\sec\theta}{2\left(\frac{\sin^{\frac{1}{2}}\theta}{\cos^{\frac{1}{2}}\theta}\right)} d\theta$

$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sin^{1/2}\theta \cos^{1/2}\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\frac{\sin 2\theta}{2}}} d\theta = \frac{1}{2}\sqrt{2}\int_0^{\pi/4} \sin^{-1/2} 2\theta\, d\theta\,,$

Put $2\theta = \phi \Rightarrow d\theta = \frac{d\phi}{2}; \quad when\ \theta = 0 \Rightarrow \phi = 0,\ \theta = \frac{\pi}{4} \Rightarrow \phi = \frac{\pi}{2}$

$I_2 = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\phi \frac{d\phi}{2} = \frac{1}{2}\frac{1}{\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\phi\, d\phi\,,$

Here $p = -\frac{1}{2} \Rightarrow \frac{p+1}{2} = \frac{1}{4}, \quad q = 0 \Rightarrow \frac{q+1}{2} = \frac{1}{2}$

$I_2 = \frac{1}{4}\frac{1}{\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right) \quad \left(\because \int_{\theta=0}^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right)$

$I_2 = \frac{1}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \ldots..(2)$

From (1) and (2) we get

$$LHS = I_1 \times I_2 = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \times \frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\pi}{4\sqrt{2}} = \text{RHS}$$

13. Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ in terms of Gamma function **[July-2009]**

Sol. let $I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} (\cot\theta)^{\frac{1}{2}} d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$

$$\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\, .\pi\sqrt{2} = \frac{\pi\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$