DIFFERENTIAL CALCULUS - 2

2.1 Taylor's and Maclaurin's series expansion for a function of one variable

The infinite series expansion of a differentiable function $y = f(x)$ about the point, $x = a$ will be a series in powers of $(x - a)$ known as *Taylor's series expansion.* In particular if $a = 0$, the series will be in ascending power of x, known as *Maclaurin's series expansion.*

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is called ***Taylor's series expansion*** of $f(x)$ about the point 'a'.

In particular, if $a = 0$ we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

This is called ***Maclaurin's series expansion*** of $f(x)$.

Note : *For convenience we shall use the following alternative notation.*
$y(x)$ for $f(x)$ and $y_1(x)$, $y_2(x)$, $y_3(x)$ . . . respectively for $f'(x), f''(x), f'''(x), \ldots$ so that we have,

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!}y_2(a) + \ldots \quad \text{[Taylor's expansion]}$$

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \ldots \quad \text{[Maclaurin's expansion]}$$

Note for solving problems (Expansion of functions)

We need to find successive derivatives of the given $y(x)$ and evaluate them at the given point $x = a$ for obtaining the Taylor's expansion and evaluate at $x = 0$ for obtaining the Maclaurin's expansion.

WORKED PROBLEMS

[1] *Obtain the Taylor's expansion of* $\log_e x$ *about* $x = 1$ *upto the term containing fourth degree and hence obtain* $\log_e (1.1)$. **[June 2017, 18, Dec 17]**

☞ We have Taylor's expansion about $x = a$ given by

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!}y_2(a) + \dots$$

By data, $y(x) = \log_e x$; $a = 1$

$y(1) = \log_e 1 = 0$. Differentiating $y(x)$ successively we get,

$$y_1(x) = \frac{1}{x} \quad \therefore y_1(1) = 1 \ ; \quad y_2(x) = \frac{-1}{x^2} \quad \therefore y_2(1) = -1$$

$$y_3(x) = \frac{2}{x^3} \quad \therefore y_3(1) = 2 \ ; \ y_4(x) = -\frac{6}{x^4} \quad \therefore y_4(1) = -6$$

Taylor's series upto fourth degree term with $a = 1$ is given by,

$$y(x) = y(1) + (x-1)y_1(1) + \frac{(x-1)^2}{2!}y_2(1) + \frac{(x-1)^3}{3!}y_3(1) + \frac{(x-1)^4}{4!}y_4(1)$$

ie., $$\log_e x = 0 + (x-1)1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6)$$

Thus, $$\boxed{\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}}$$

We shall now substitute $x = 1.1$ to obtain $\log(1.1)$

Also $$\log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.0953$$

Thus, $\boxed{\log_e(1.1) = 0.0953}$

[3] *Obtain Taylor's series expansion of* log (cos x) *about the point* $x = \pi/3$ *up to the fourth degree term.*

[Dec 201

Taylor's expansion of $y(x)$ about $x = \pi/3$ is given by

$$y(x) = y(\pi/3) + (x - \pi/3)\, y_1(\pi/3) + \frac{(x-\pi/3)^2}{2!}\, y_2(\pi/3)$$

$$+ \frac{(x-\pi/3)^3}{3!}\, y_3(\pi/3) + \frac{(x-\pi/3)^4}{4!}\, y_4(\pi/3) + \ldots \quad \ldots (1$$

Let, $y(x) = \log(\cos x)$ $\quad \therefore \quad y(\pi/3) = \log(1/2) = -\log 2$

$$y_1 = \frac{1}{\cos x} \cdot -\sin x$$

ie., $y_1 = -\tan x \qquad \therefore \quad y_1(\pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$y_2 = -\sec^2 x = -(1+\tan^2 x)$

ie., $y_2 = -(1+y_1^2) \qquad \therefore \quad y_2(\pi/3) = -[1+(\sqrt{3})^2] = -4$

$y_3 = -2y_1 y_2 \qquad \therefore \quad y_3(\pi/3) = -2 \cdot -\sqrt{3} \cdot -4 = -8\sqrt{3}$

$y_4 = -2[y_1 y_3 + y_2^2] \qquad \therefore \quad y_4(\pi/3) = -2[-\sqrt{3} \cdot -8\sqrt{3} + 16] = -80$

Substituting these values in (1) we have,

$$\log(\cos x) = -\log 2 - (x-\pi/3)\sqrt{3} - \frac{(x-\pi/3)^2}{2} \cdot 4 - \frac{(x-\pi/3)^3}{6} \cdot 8\sqrt{3} - \frac{(x-\pi/3)^4}{24} \cdot 80 + \ldots$$

Thus,

$$\boxed{\log(\cos x) = -\log 2 - \sqrt{3}(x-\pi/3) - 2(x-\pi/3)^2 - \frac{4}{\sqrt{3}}(x-\pi/3)^3 - \frac{10}{3}(x-\pi/3)^4}$$

Note : ***Similar problem***

Expand sin x in power of $\left(x - \frac{\pi}{2}\right)$ *upto fourth degree terms.* **[Dec 2015, June16]**

☞ We need to evaluate $y(x) = \sin x$ and its successive derivatives at $x = \pi/2$.

They are, $y = (\pi/2) = 1$, $y_1(\pi/2) = 0$, $y_2(\pi/2) = -1$, $y_3(\pi/2) = 0$, $y_4(\pi/2) = 1$

Thus,

$$\boxed{y(x) = \sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} + \cdots}$$

[5] *Expand $e^{\sin x}$ as Maclaurin's series upto the terms containing x^4.* **[Dec 2017]**

☞ We have Maclaurin's expansion,

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots$$

Let, $y = e^{\sin x}$ $\quad\therefore\ y(0) = e^0 = 1$

$y_1 = e^{\sin x}\cos x$ or $y_1 = y\cos x$ $\quad\therefore\ y_1(0) = y(0)\cdot\cos 0 = 1$

$y_2 = -y\sin x + \cos x\cdot y_1$ $\quad\therefore\ y_2(0) = 0 + 1 = 1$

$y_3 = -(y\cos x + y_1\sin x) + (\cos x\, y_2 - y_1\sin x)$

$= -y_1 - 2y_1\sin x + \cos x\cdot y_2$ $\quad\therefore\ y_3(0) = -1 - 0 + 1 = 0$

$y_4 = -y_2 - 2(y_1\cos x + \sin x\, y_2) + (\cos x\, y_3 - \sin x\, y_2)$

$= -y_2 - 2y_1\cos x - 3\sin x\, y_2 + \cos x\cdot y_3$

$y_4(0) = -1 - 2 - 0 + 0 = -3$ $\quad\therefore\ y_4(0) = -3$.

Thus by substituting these values in the expansion of $y(x)$ we get,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

[7] *Expand* log (sec *x*) *upto the term containing* x^6 *using Maclaurin's seri*

or [Dec 20

Expand log (sec *x*) *in ascending powers of x upto the first three non vanish terms.*

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots$$

$y = \log(\sec x)$ $\therefore \; y(0) = \log 1 = 0$

$y_1 = \dfrac{\sec x \tan x}{\sec x}$ or $y_1 = \tan x$ $\therefore \; y_1(0) = 0$

$y_2 = \sec^2 x$ $\therefore \; y_2(0) = 1$

Now, $y_2 = 1 + \tan^2 x = 1 + y_1^2$

Differentiating w.r.t. x successively we have,

$y_3 = 2y_1 y_2$ $\therefore \; y_3(0) = 0$

$y_4 = 2(y_1 y_3 + y_2^2)$ $\therefore \; y_4(0) = 2$

$y_5 = 2(y_1 y_4 + y_2 y_3 + 2y_2 y_3) = 2y_1 y_4 + 6y_2 y_3$ $\therefore \; y_5(0) = 0$

$y_6 = 2(y_1 y_5 + y_2 y_4) + 6(y_2 y_4 + y_3^2)$

ie., $y_6 = 2y_1 y_5 + 8y_2 y_4 + 6y_3^2$ $\therefore \; y_6(0) = 16$

Substituting these values in the expansion of $y(x)$ we get,

$$\log(\sec x) = \frac{x^2}{2} * 1 + \frac{x^4}{24} * 2 + \frac{x^6}{720} * 16 = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$$

[11] *Expand,* tan (π/4 + x) *upto the fourth degree terms.*

☞ $$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)\ldots$$

Let, $y = \tan(\pi/4 + x)$ $\therefore\ y(0) = \tan(\pi/4) = 1$

$y_1 = \sec^2(\pi/4 + x) = 1 + y^2$ $\therefore\ y_1(0) = 2$

$y_2 = 2yy_1$ $\therefore\ y_2(0) = 4$

$y_3 = 2(yy_2 + y_1^2)$ $\therefore\ y_3(0) = 2(4+4) = 16$

$y_4 = 2(yy_3 + 3y_1y_2)$ $\therefore\ y_4(0) = 2(16+24) = 80$

Substituting these values in the expansion of $y(x)$ we have,

$$\tan(\pi/4 + x) = 1 + x \cdot 2 + \frac{x^2}{2} \cdot 4 + \frac{x^3}{6} \cdot 16 + \frac{x^4}{24} \cdot 80$$

Thus, $$\boxed{\tan(\pi/4 + x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4}$$

[13] *Obtain the Maclaurin's expansion of* log $(1 + e^x)$ *as far as the fourth degree terms.*

[June 201

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots$$

Let, $y = \log(1 + e^x)$ $\therefore$ $y(0) = \log_e 2$

$$y_1 = \frac{e^x}{1+e^x} \qquad \therefore \; y_1(0) = \frac{1}{2}$$

ie., $(1 + e^x) y_1 = e^x$ ... (1)

Differentiating w.r.t x we get,

$(1 + e^x) y_2 + e^x y_1 = e^x$ ... (2)

At $x = 0$, $2y_2(0) + 1/2 = 1$ $\therefore$ $y_2(0) = 1/4$

Differentiating (2) w.r.t x we get,

$(1 + e^x) y_3 + 2 e^x y_2 + e^x y_1 = e^x$ ... (3)

At $x = 0$, $2y_3(0) + 1/2 + 1/2 = 1$ $\therefore$ $y_3(0) = 0$

Differentiating (3) w.r.t x we get,

$(1 + e^x) y_4 + 3 e^x y_3 + 3 e^x y_2 + e^x y_1 = e^x$ ... (4)

At $x = 0$, $2y_4(0) + 3/4 + 1/2 = 1$ $\quad \therefore \; y_4(0) = -1/8$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\boxed{\log(1+e^x) = \log_e 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}}$$

Note : ***Similar problem***

Expand $\frac{e^x}{1+e^x}$ *using Maclaurin's series upto and including third degree terms.*

[Dec 2016]

☞ The given $y = \frac{e^x}{1+e^x}$ is same as y_1 of the earlier problem. $y(0) = 1/2$ and proceeding on the same lines the other values will be

$$y_1(0) = 1/4,\; y_2(0) = 0,\; y_3(0) = -1/8$$

Thus the required expansion is given by

$$\boxed{\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48}}$$

[14] *Expand* log (1 + cos x) *by Maclaurin's series upto the term containing* x^4.

[June 2018]

☞ Maclaurin's series is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots$$

Consider, $y = (x) = \log(1 + \cos x)$; $y(0) = \log_e 2$

We prefer to simplify the given functions

$$y = \log(1 + \cos x) = \log[2\cos^2(x/2)]$$

ie., $y = \log 2 + 2\log\cos(x/2)$ $\quad \therefore \; y(0) = \log_e 2$

Now, $y_1 = -\tan(x/2)$ $\quad \therefore \; y_1(0) = 0$

$y_2 = -(1/2)\sec^2(x/2)$ $\quad \therefore \; y_2(0) = -1/2$

Also, $y_2 = -\frac{1}{2}[1 + \tan^2(x/2)] = -\frac{1}{2}(1 + y_1^2)$

$\therefore \quad y_3 = -\frac{1}{2}(2y_1y_2) = -y_1y_2$ $\quad \therefore \; y_3(0) = 0$

$y_4 = -y_1y_3 - y_2^2$ $\quad \therefore \; y_4(0) = -1/4$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\log_e(1+\cos x) = \log_e 2 - \frac{x^2}{4} - \frac{x^4}{96}$$

[15] *Obtain the Maclaurin's expansion of a^x.*

☞ $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots$

Let, $y = a^x$ $\therefore$ $y(0) = 1$

$y_1 = a^x \log a = y \log a$ $\therefore$ $y_1(0) = \log a$

$y_2 = y_1 \log a$ $\therefore$ $y_2(0) = (\log a)^2$

$y_3 = y_2 \log a$ $\therefore$ $y_3(0) = (\log a)^3$ and so on.

Thus by substituting these values in the expansion of $y(x)$ we get,

$$a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$$

Note : *Maclaurin's expansions of the functions* sin x, cos x, sin h x, cosh x, e^x *can be found easily and it is advisible to remember them. They are as follows.*

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iii) $\sin h\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$

(iv) $\cos h\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

(v) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$